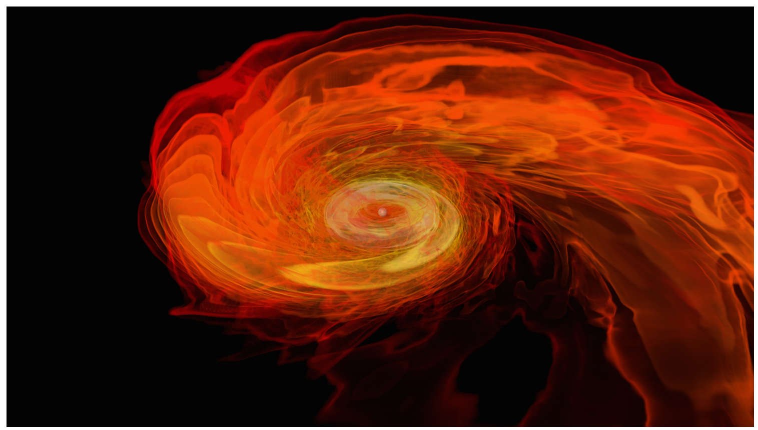
Lecture 3

Anatomy of a merger

St Andrews summer school 2017



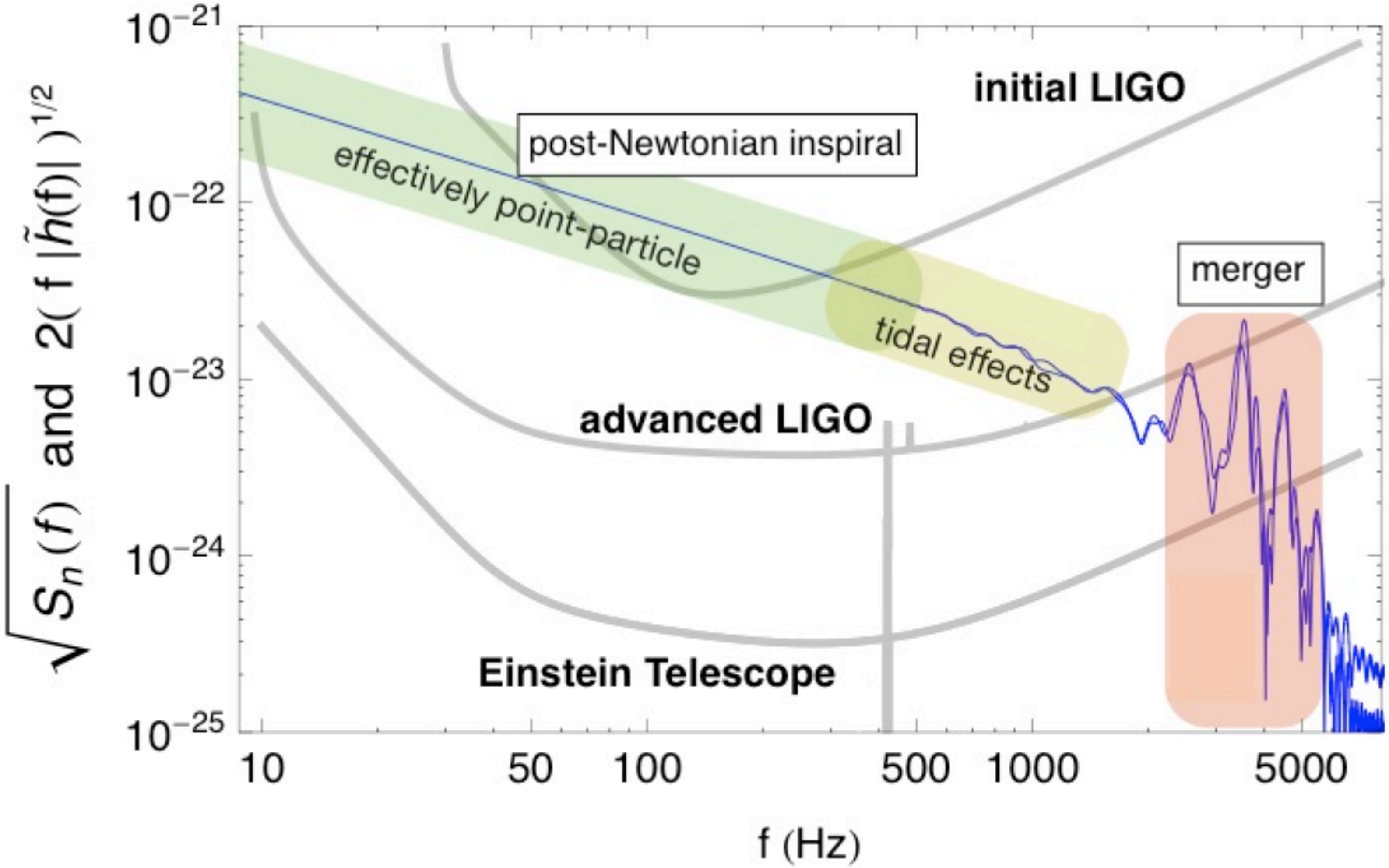
Nils Andersson

na@maths.soton.ac.uk

STAG RESEARCH CENTRE

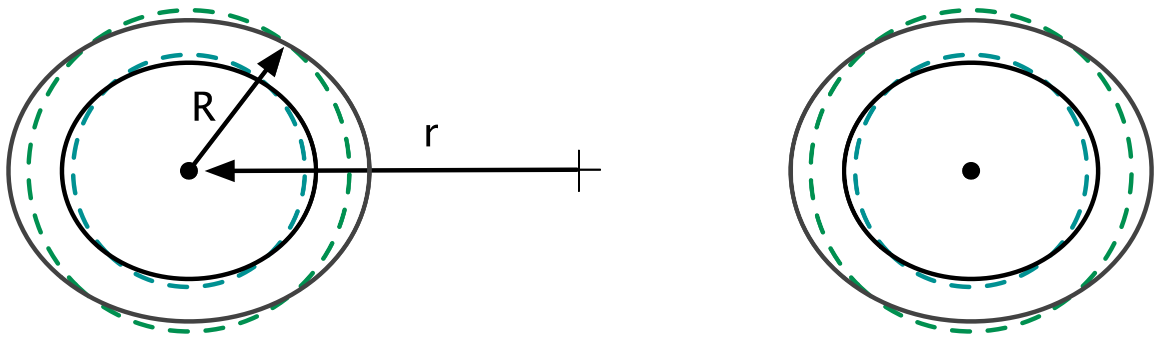
UNIVERSITY OF Southampton

tidal deformation ➔ resonances ➔ merger oscillations



The deviation from point-mass dynamics (the fact that we are dealing with fluid bodies!) becomes important at the late stages of binary inspiral.

Each star is deformed by the tidal interaction from its companion.



The tidal effect will not change the GW phasing much – this is difficult; e.g. 10^{46} erg at 100 Hz leads to shift of only 10^{-3} radians - but the star's deformability, encoded in the so-called Love number, may lead to a measurable secular effect.

For each star, we have

$$\lambda = \frac{2}{3} k_2 R^5 = \frac{Q}{E} = \frac{\text{quadrupole deformation}}{\text{strength of tidal field}}$$

Given that the masses can be extracted from the chirp, observations may allow us to constrain the compressibility of the equation of state.

Note: If the radius can be inferred with an accuracy <5%, we would do better than any upcoming nuclear physics experiments (but not NICER?)

Let us take a closer look at the tidal problem.

The impact of the tide from a binary companion is obtained from the perturbed Euler equation;

$$\partial_t^2 \xi^i + \frac{1}{\rho}\nabla^i \Delta p - \frac{\Delta\rho}{\rho^2}\nabla^i p + \nabla^i \Delta\Phi = -\nabla^i \chi$$

where the tidal potential is given by

$$\chi = -\frac{GM'}{|\boldsymbol{r} - \boldsymbol{D}(t)|} = -GM' \sum_{l\geq 2} \sum_{m=-l}^{l} \frac{W_{lm} r^l}{D^{l+1}(t)} Y_{lm} e^{-im\psi(t)}$$

where $D(t)$ is the radius of the orbit and $\psi(t)$ is the phase.

There are two contributions:

- for $m=0$ we have a static contribution (evolving on the inspiral timescale). This encodes the star's tidal compressibility in the form of the Love number
- For $m=\pm 2$ we have a time-dependent tidal driving which may become resonant with the star's natural modes of oscillation.

To work out the Love number, we consider static perturbations. That is, we have

$$\frac{1}{\rho}\nabla^i\delta p - \frac{1}{\rho^2}\delta\rho\nabla^i p + \nabla^i\delta\Phi = -\nabla^i\chi$$

Expanding in spherical harmonics (with m=0), we have an effective gravitational potential

$$U_l = \Phi_l + \chi_l$$

which satisfies

$$r^2 U_l'' + 2rU_l' - l(l+1)U_l = 4\pi G r^2 \rho_l$$

Combining radial and angular components of the Euler equations we find that

$$\rho' p_l = p'\rho_l$$

Using this we can reduce the problem to a single ODE;

$$r^2 U_l'' + 2rU_l' + \left[\frac{4\pi G r^2\rho^2}{p\Gamma_\beta} - l(l+1)\right]U_l = 0$$

where

$$\delta p = \frac{p\Gamma_\beta}{\rho}\delta\rho \qquad\qquad \Gamma_\beta = \frac{\rho}{p}\left(\frac{\partial p}{\partial\rho}\right)_{\mathrm{eq}}$$

Note: The results imply that the fluid remains in chemical equilibrium throughout the inspiral. This is probably not realistic.

Match to the external solution (at the surface of the star), where

$$\Phi_l = \frac{4\pi G}{2l+1}\frac{I_l}{r^{l+1}} \qquad \chi_l = \frac{4\pi}{2l+1} d_l r^l \quad \Rightarrow \quad U_l = \frac{4\pi}{2l+1}\left(\frac{GI_l}{R^{l+1}} + d_l R^l\right)$$

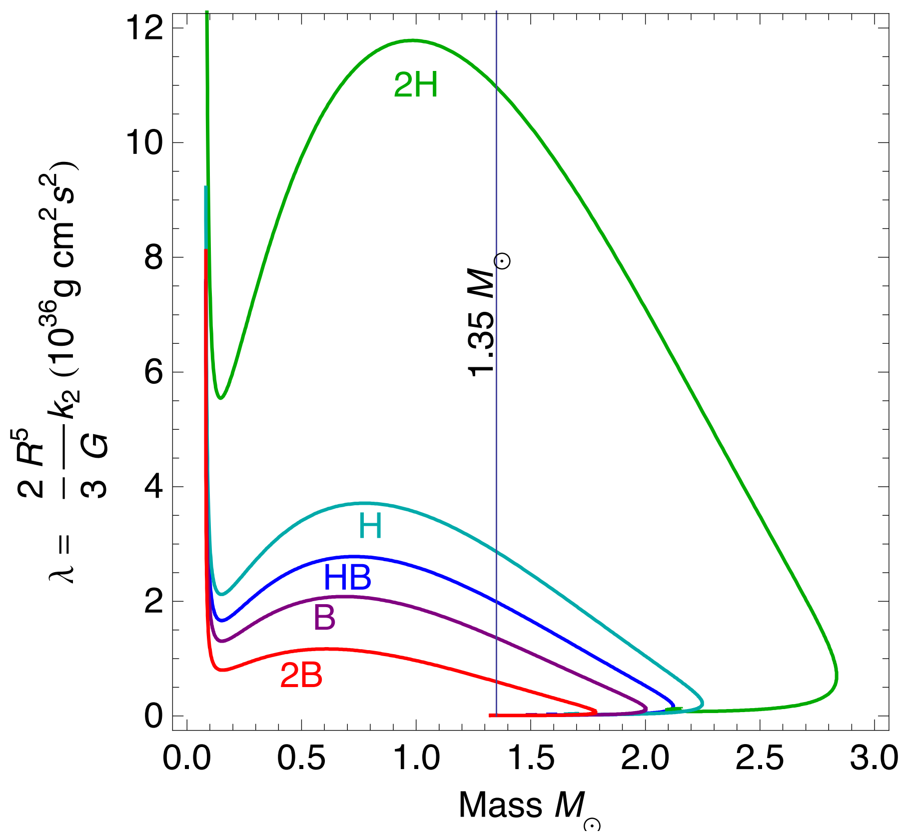
And extract the Love number k_l from

$$GI_l = 2k_l R^{2l+1} d_l$$

If we want quantitative results, we need to do the relativistic calculation. However, it proceeds along similar lines.

Results show the variation of the deformability for representative "realistic" equations of state.

Suggests that we may be able to use observations to distinguish different models, but... as the tidal imprint is weak, it may require the combination of many signals.



[Read et al 2013]

Turning to the time-dependent part of the problem, the tide can lead to resonances with many of star's of oscillation. However, typically, the coupling tends to be weak and the resonances do not have significant effect.

The main interaction may be with the star's f-mode.

In order to understand this, let us sketch the derivation of the f-mode for an incompressible fluid. The starting point is the perturbed Euler equation (but with $\delta\rho=0$):

$$\partial_t \delta \boldsymbol{v} + \frac{1}{\rho}\nabla\delta p + \nabla\delta\Phi = 0 \qquad\qquad \nabla\cdot\boldsymbol{v} = 0$$

As the flow must be irrotational, we have

$$\delta\boldsymbol{v} = \nabla\chi \quad \Rightarrow \quad \partial_t\chi + \frac{1}{\rho}\delta p + \delta\Phi = D = \text{ constant}$$

where we have D=0, as the left-hand side vanishes at the centre of the star.

Find that: $\nabla^2\chi = 0 \qquad \nabla^2\delta\Phi = 0 \qquad \nabla^2\delta p = 0$

Expanding in spherical harmonics (with m=0, without loss of generality), with harmonic time dependence:

$$\chi = a_l r^l Y_l^m$$

$$\delta\Phi = b_l r^l Y_l^m \quad \Rightarrow \quad i\omega a_l + \frac{c_l}{\rho} + b_l = 0$$

$$\delta p = c_l r^l Y_l^m$$

Finally, we need to impose boundary conditions at the surface. For the pressure we need

$$\Delta p = \delta p + \xi^r \frac{dp}{dr} = 0 \quad \Rightarrow \quad c_l = \frac{4\pi G \rho^2 l}{3 i \omega} a_l$$

If we also use the Cowling approximation (set $b_l = 0$), then we arrive at the mode frequency:

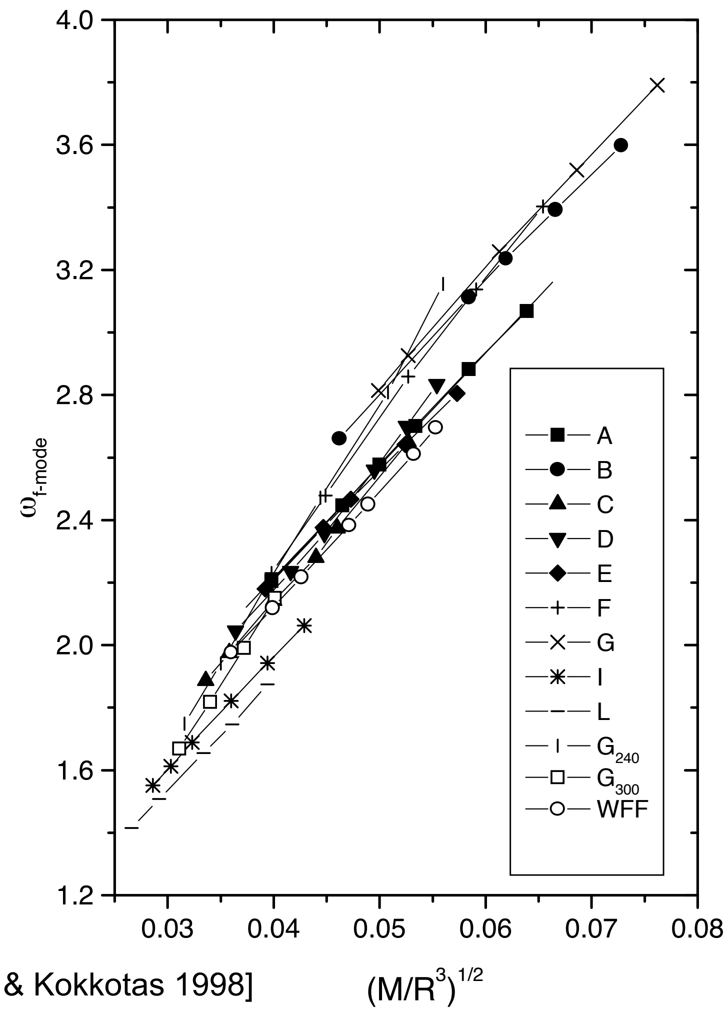
$$\omega^2 = \frac{4\pi G \rho l}{3}$$

Main lesson: The f-mode frequency scales with the average density of the star.

Even though our calculation involved a number of simplifications, this scaling is robust.

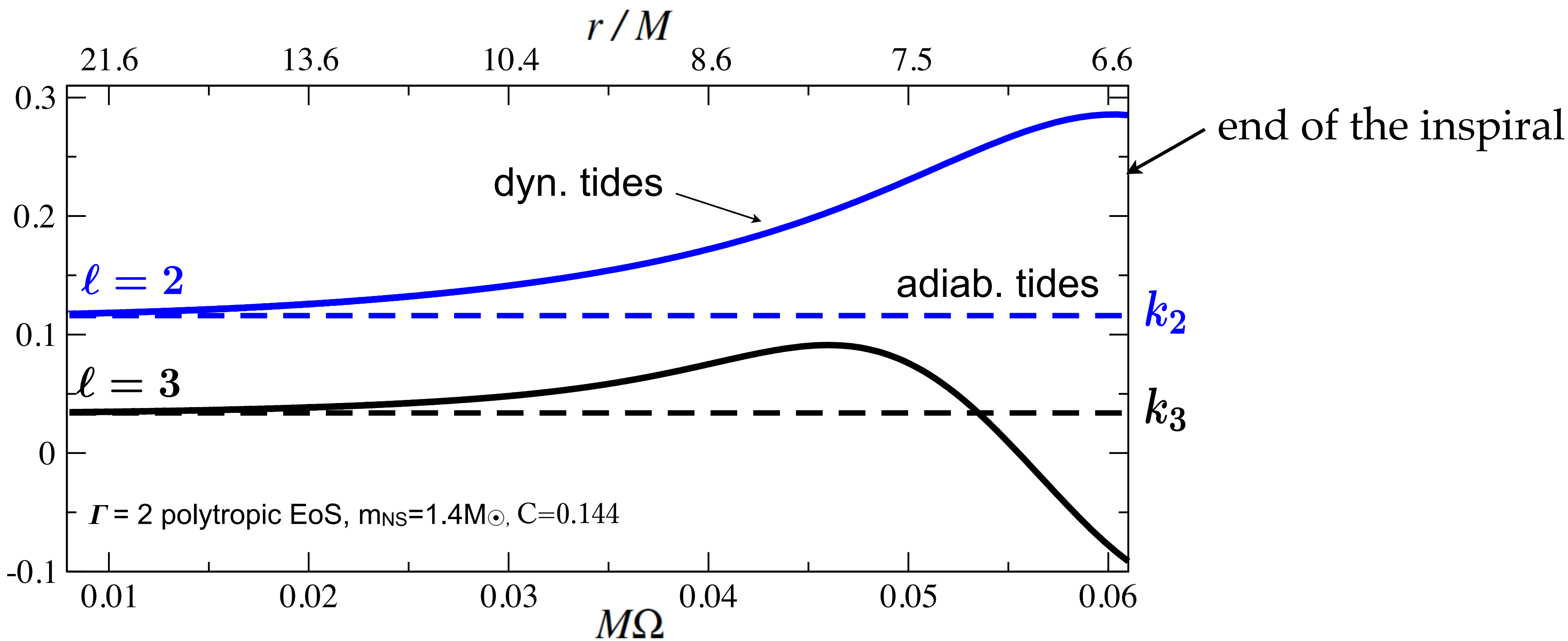
Leads to the idea of astero-seismology, where we try to extract the parameters of the star from observed data.

QPOs in the tails of magnetar flares have already allowed us to try this out.



[Andersson & Kokkotas 1998]

The f-mode frequency is generally too high to become resonant with the tidal driving, but recent work suggests that the associated dynamical contribution to the tide is nevertheless significant.



[Hinderer et al 2016]

The final merger involves violent dynamics that can only be studied using full nonlinear simulations.

This involves working with a foliation of spacetime (rather than the fibration we used in the derivation of the relativistic fluid equations.

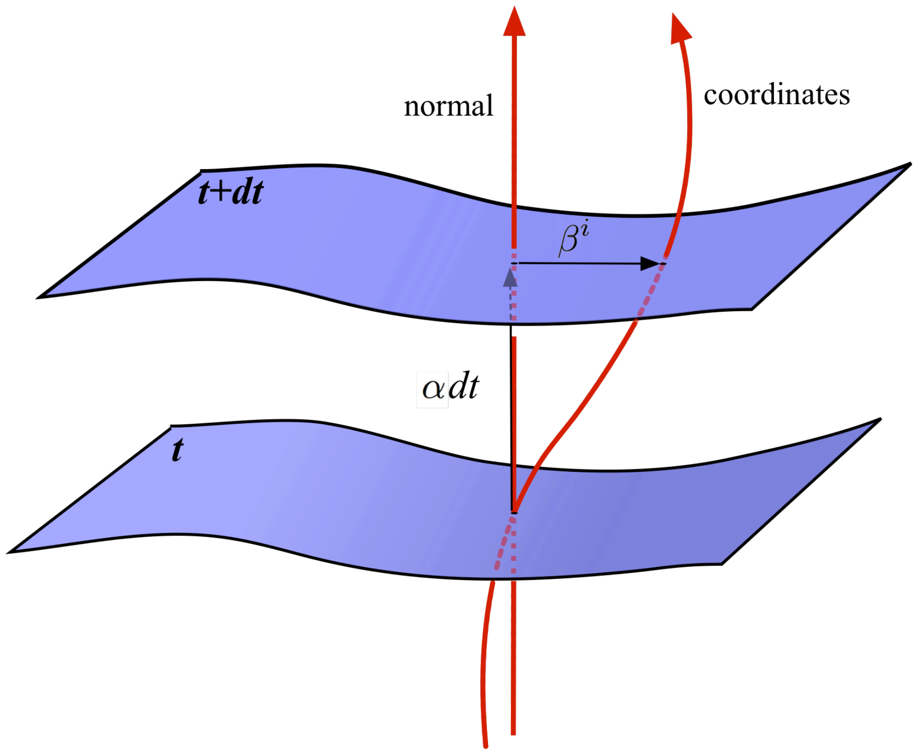
As in the vacuum case, one introduces a family of spacelike hypersurfaces as level surfaces of a scalar "time" t. The normal to these surfaces is

$$N_\mu = -\alpha \nabla_\mu t$$

and

$$t^\mu = \alpha N^\mu + \beta^\mu$$

where α is the lapse and β^μ the shift.



The velocity of a fluid is now given by

$$u^\mu = W(N^\mu + v^\mu) \qquad N_\mu v^\mu = 0$$

$$W = -N_\mu u^\mu = \alpha u^t = (1 - v_i v^i)^{-1/2}$$

The conserved (baryon) number flux now follows from

$$\nabla_\mu(nu^\mu) = \nabla_\mu[Wn(N^\mu + v^\mu)] = 0$$

$$\Rightarrow \quad \partial_t\left(\gamma^{1/2}\hat{n}\right) + D_i\left[\gamma^{1/2}\hat{n}(\alpha v^i - \beta^i)\right] = 0$$

where γ is the determinant of the spatial metric, D_i is the covariant derivative in the hypersurface and the number density measured by the Eulerian observer is

$$\hat{n} = -N_\mu n u^\mu = nW$$

Note: the 3+1 version resembles the usual Newtonian result.

Energy/momentum conservation is more intricate. In the case of a perfect fluid:

foliation

$$T^{\mu\nu} = \rho N^\mu N^\nu + 2N^{(\mu}S^{\nu)} + S^{\mu\nu}$$

$$\rho = N_\mu N_\nu T^{\mu\nu} = \varepsilon W^2 - p\left(1 - W^2\right)$$

$$S^i = -\gamma^i_\mu N_\nu T^{\mu\nu} = (p+\varepsilon)\, v^i$$

$$S^{ij} = \gamma^i_\mu \gamma^j_\nu T^{\mu\nu} = p\gamma^{ij} + (p+\varepsilon)\, W^2 v^i v^j$$

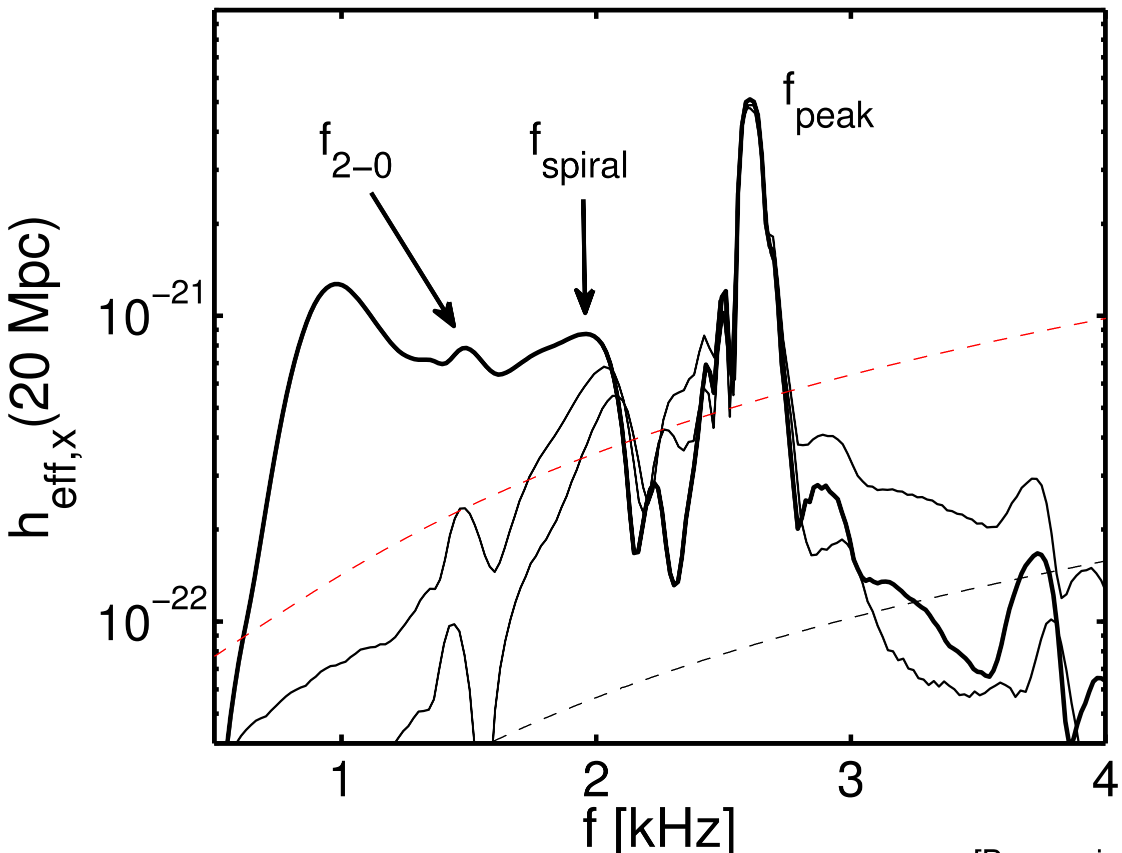
fibration

$$T^{\mu\nu} = (p+\varepsilon)u^\mu u^\nu + pg^{\mu\nu}$$

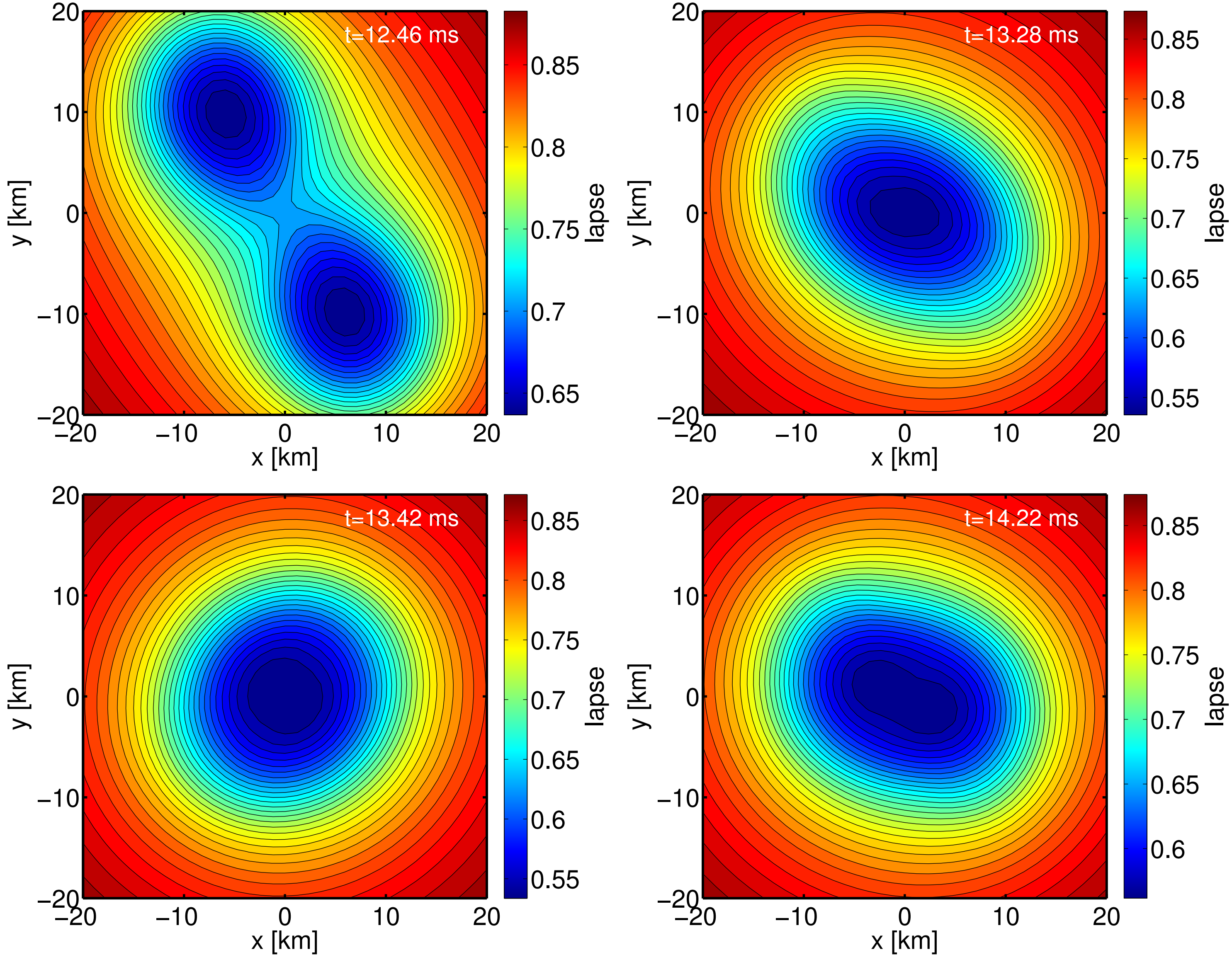
You need to revert to primitive (fluid) variables in order to use EoS data! Tricky if you include more physics (MHD).

Even though the merger involves violent dynamics (and complex physics) we are learning that there are “robust” features.

In particular, there are peaks in the merger spectrum that can be associated with stellar oscillation. The “f-mode” is particularly prominent.

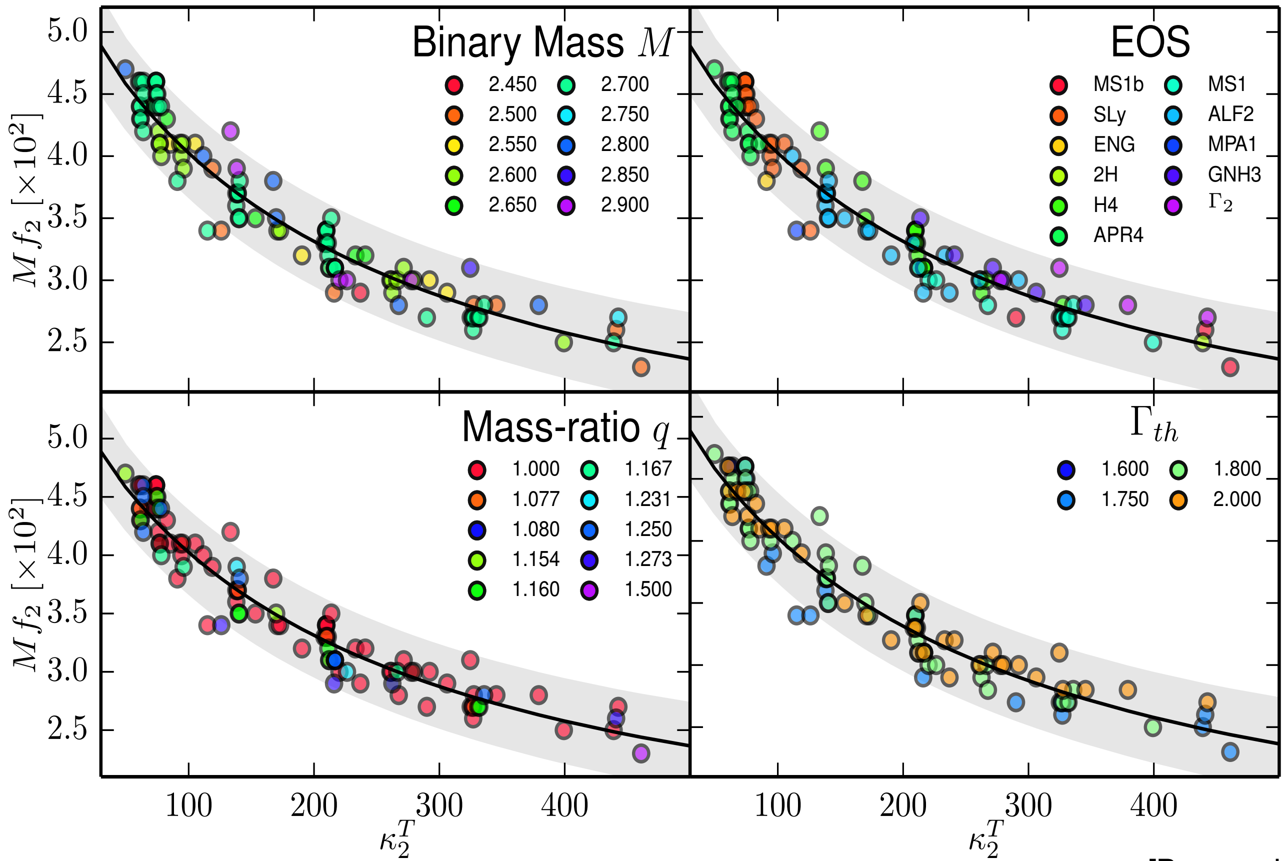


[Bauswein et al]



[Bauswein et al]

... and there is a “surprising” correlation with the tidal deformability!



[Bernuzzi & Dietrich]

This should be good news for effort to extract information about the EoS.