Detecting gravitational waves II: How well can we measure them?

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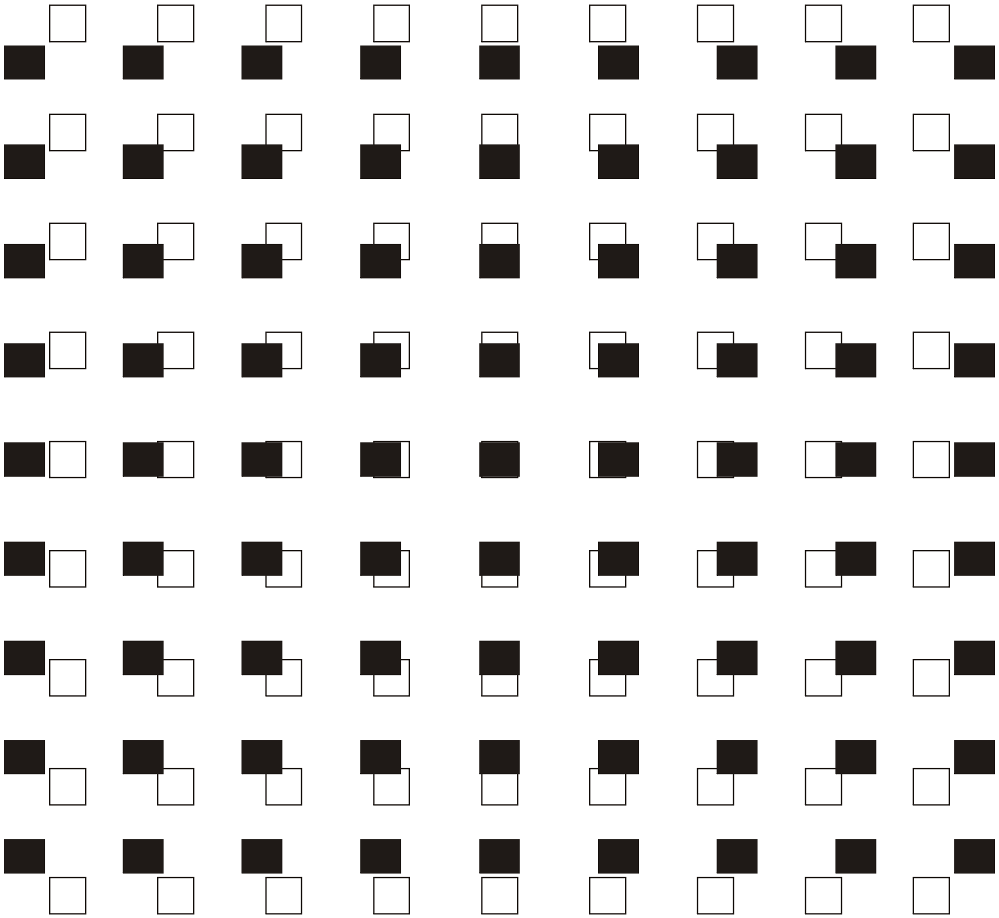
Outline

1. The challenge of gravitational wave detection
2. Core measurement sensitivity: shot noise
3. External mechanical noise: seismic noise
4. Internal mechanical noise: thermal noise

Gravitational wave: a transverse quadrupolar strain

strain amplitude:
$h = 2\Delta L/L$



Gravitational waveform lets you read out source dynamics

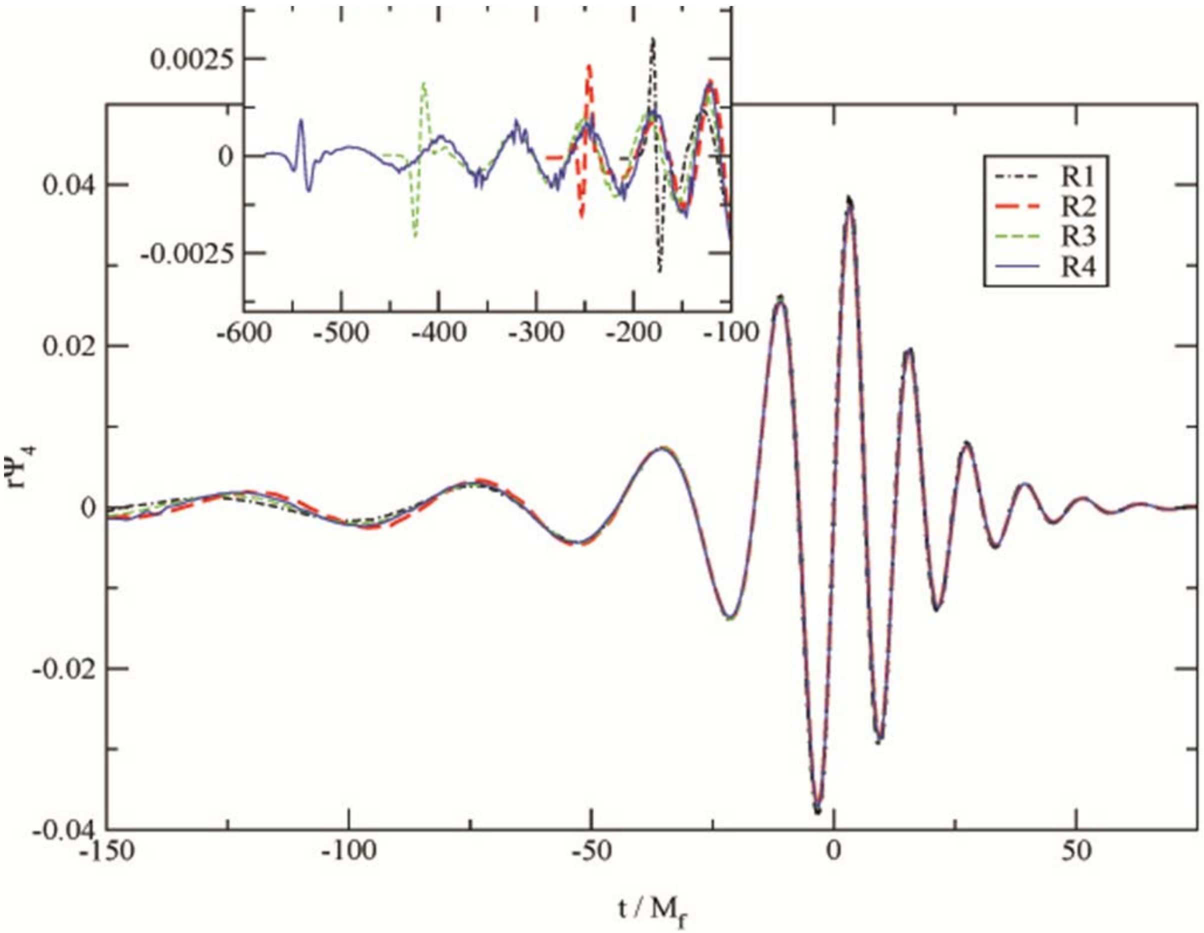
The evolution of the mass distribution can be read out from the gravitational waveform:

$$h_{\mu\nu}(t) = \frac{1}{R}\frac{2G}{c^4}\ddot{I}_{\mu\nu}(t - R/c)$$

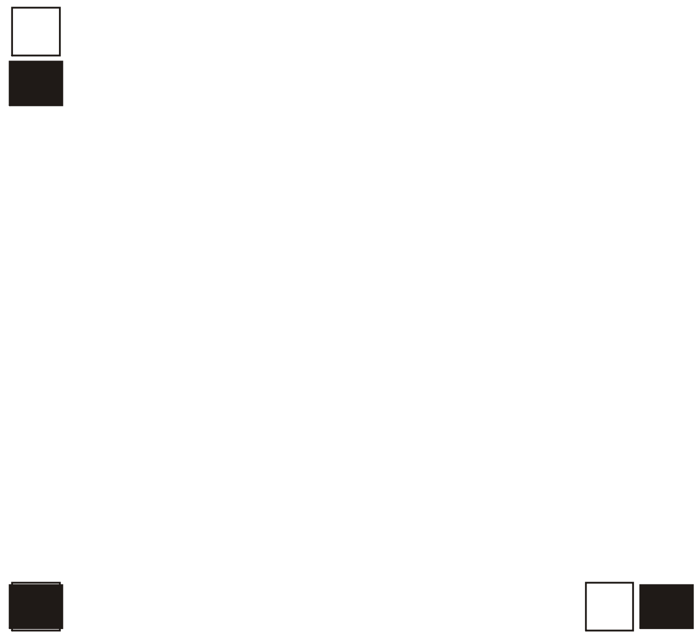
Coherent relativistic motion of large masses can be directly observed from the waveform!

$$I_{\mu\nu} \equiv \int dV \left(x_\mu x_\nu - \delta_{\mu\nu} r^2 / 3 \right) \rho(r).$$

Gravitational waveform = oscillation pattern of test masses



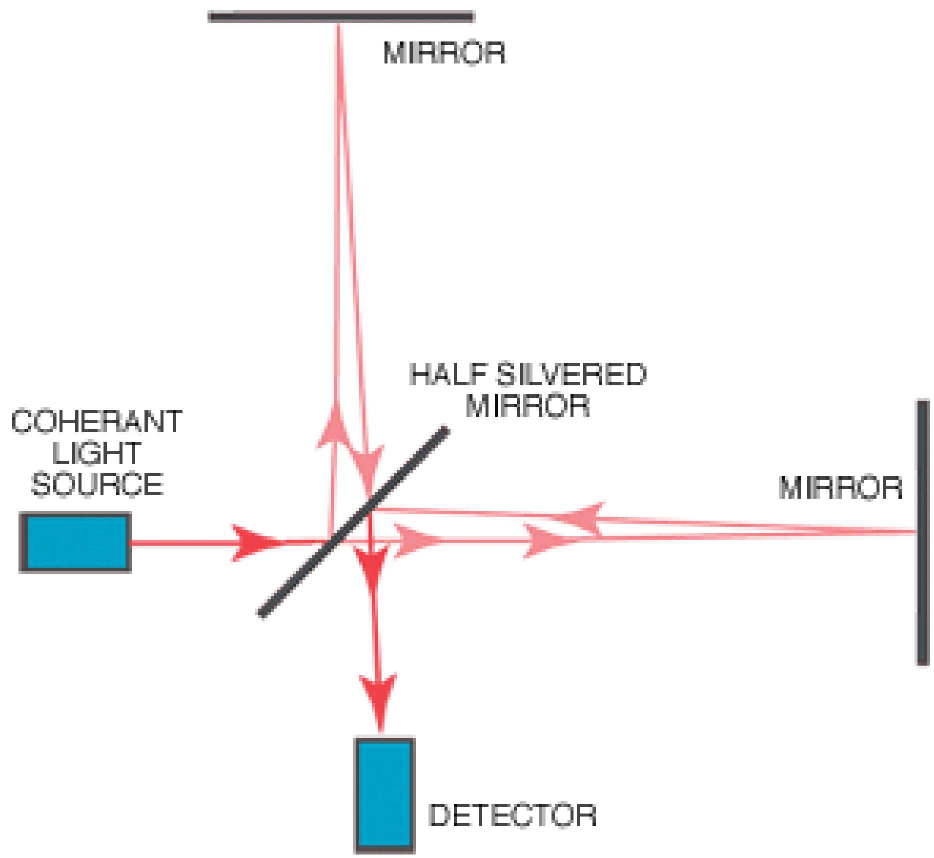
A more modern detection strategy



Tidal character of wave argues for test masses as far apart as practicable. Perhaps masses hung as pendulums, kilometers apart.

Sensing relative motions of distant free masses

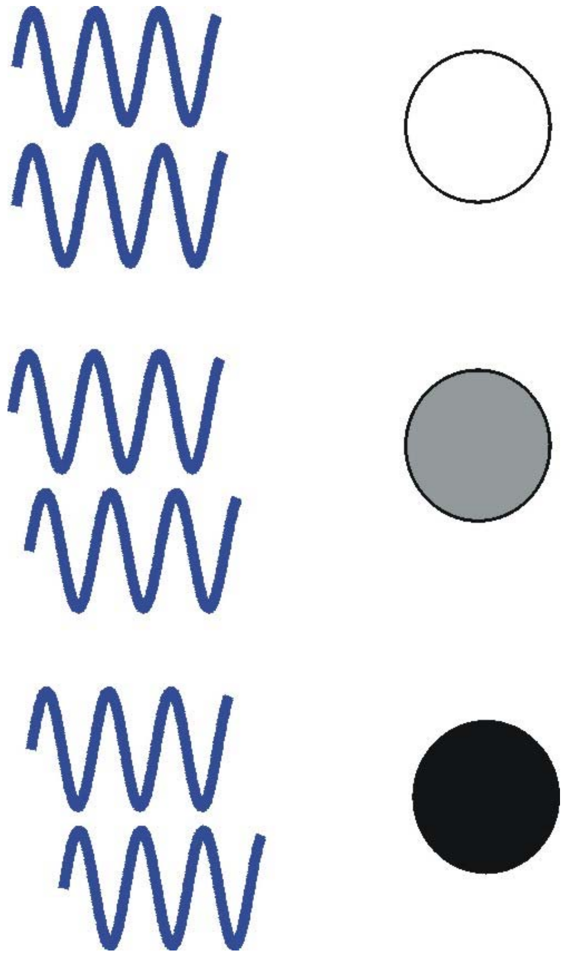
Michelson interferometer



A length-difference-to-brightness transducer

Wave from x arm.

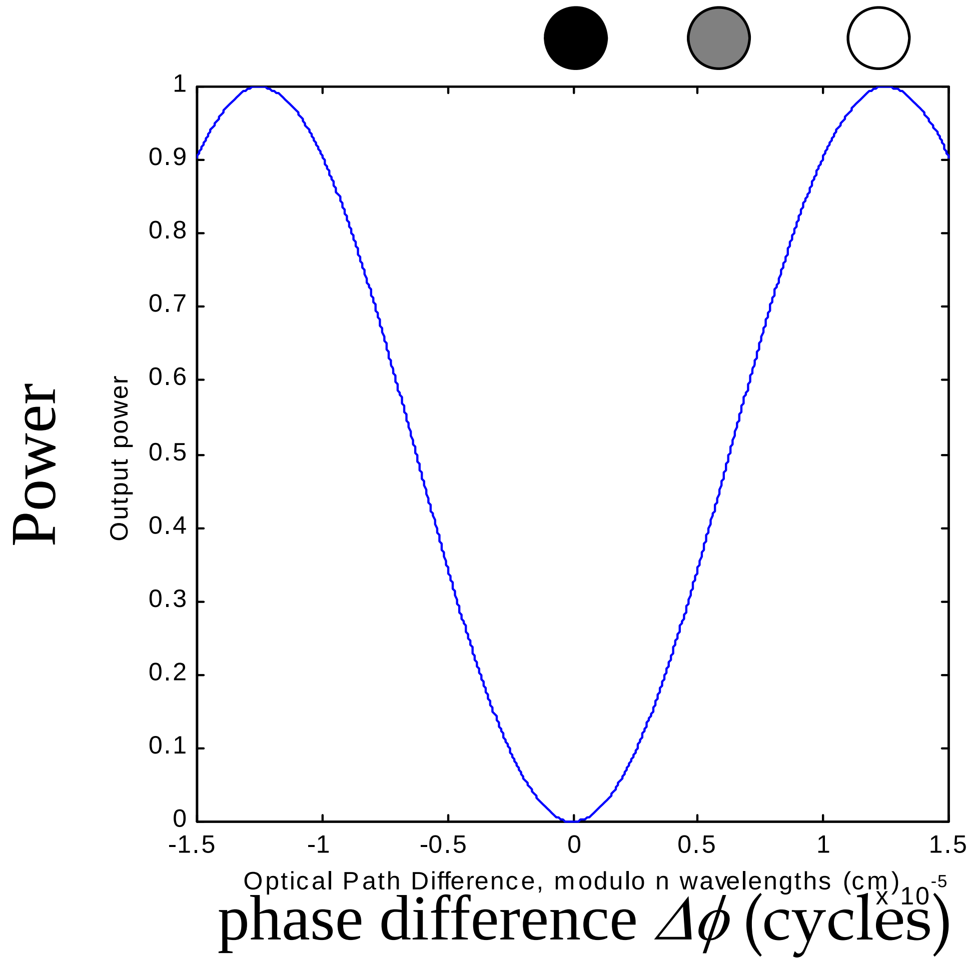
Wave from y arm.



Light exiting from beam splitter.

As relative arm lengths change, interference causes change in brightness at output.

Interferometer output vs. arm length difference



Gravity wave detectors

Need:

- A set of test masses,
- Instrumentation sufficient to see tiny motions,
- Isolation from other causes of motions.

Challenge:

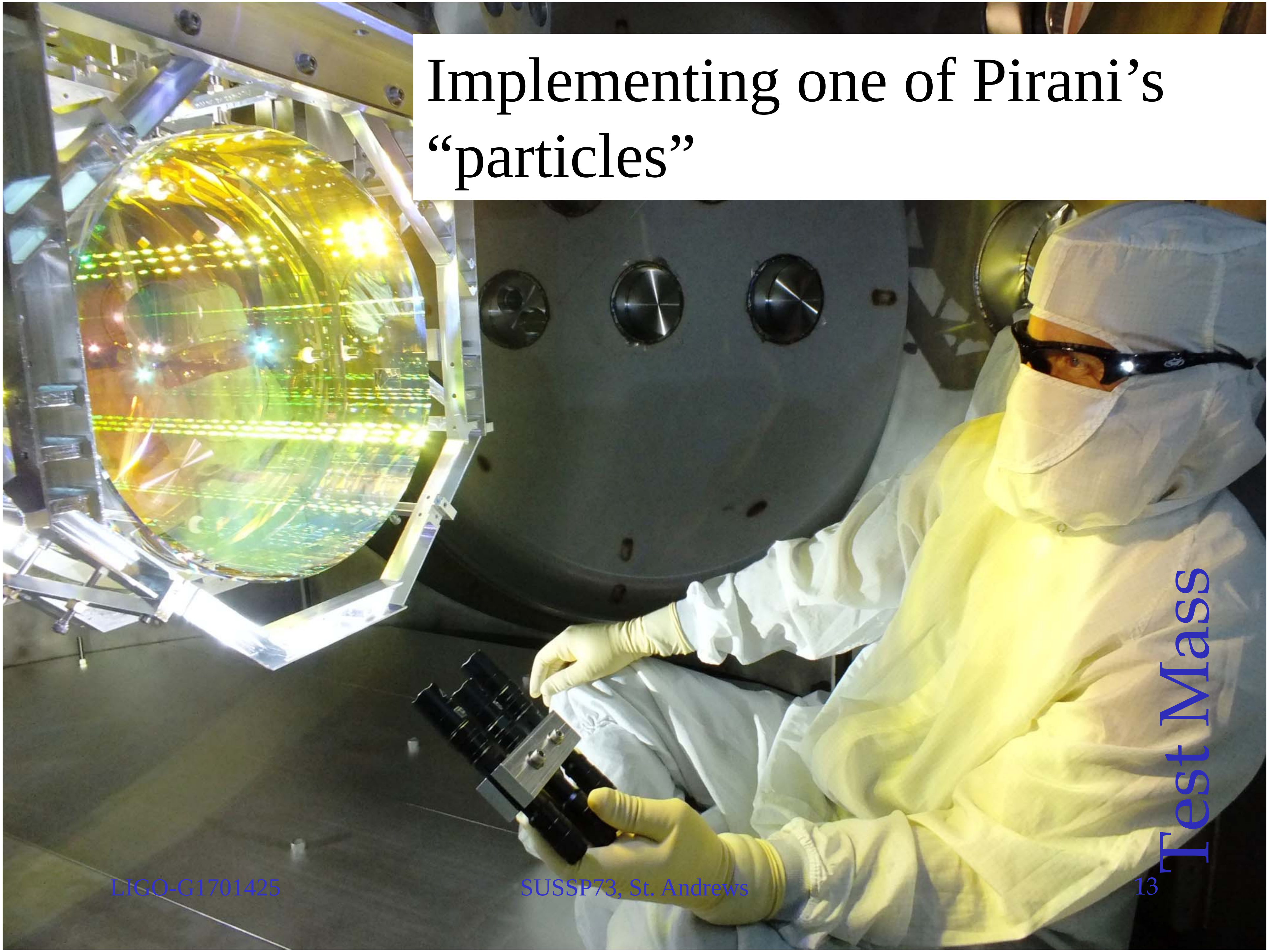
Best astrophysical estimates had long predicted fractional separation changes of at most 1 part in 10^{21}, or less.

(We now know that those estimates were correct.)

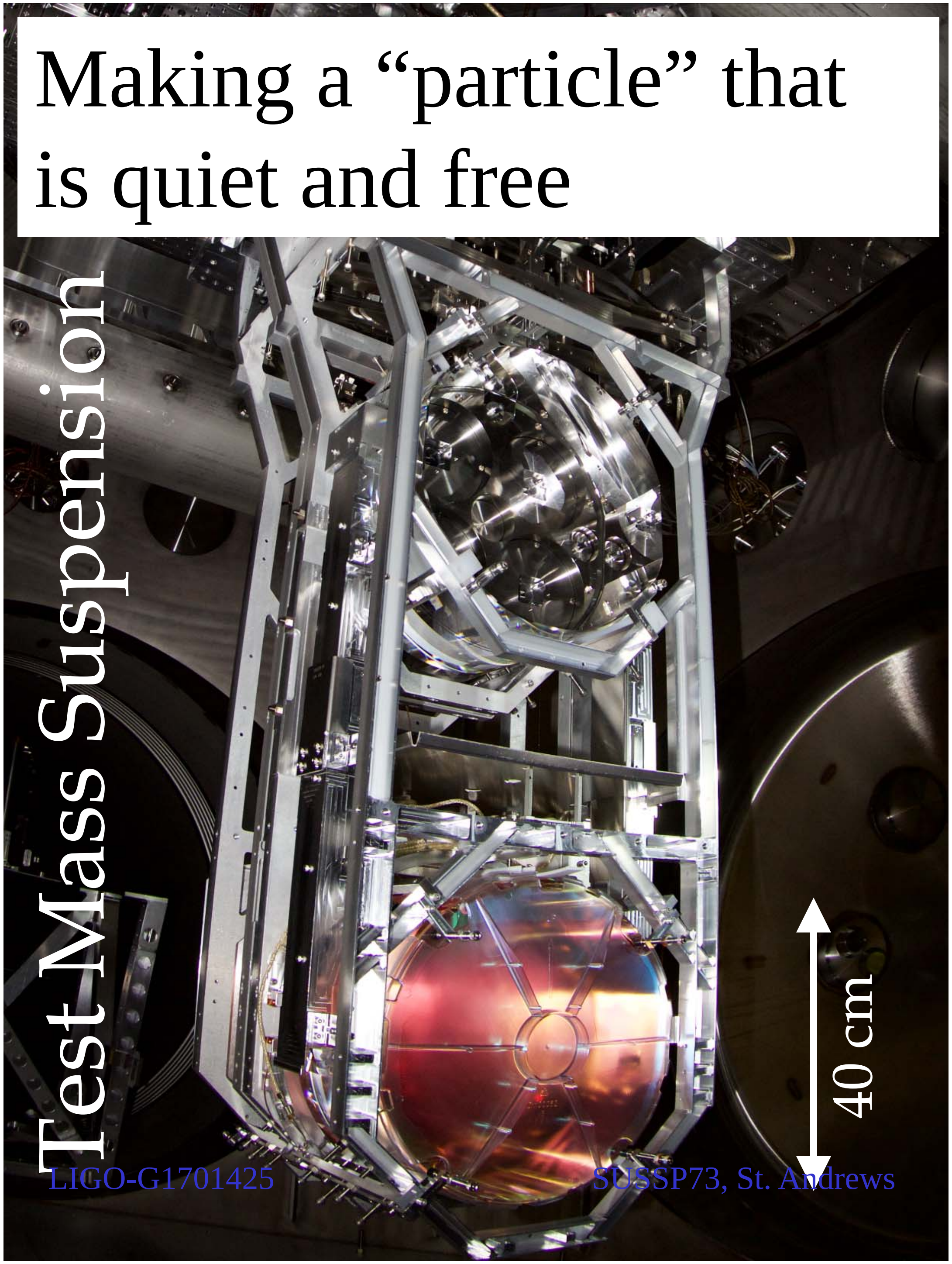
LIGO Observatory at Livingston, LA

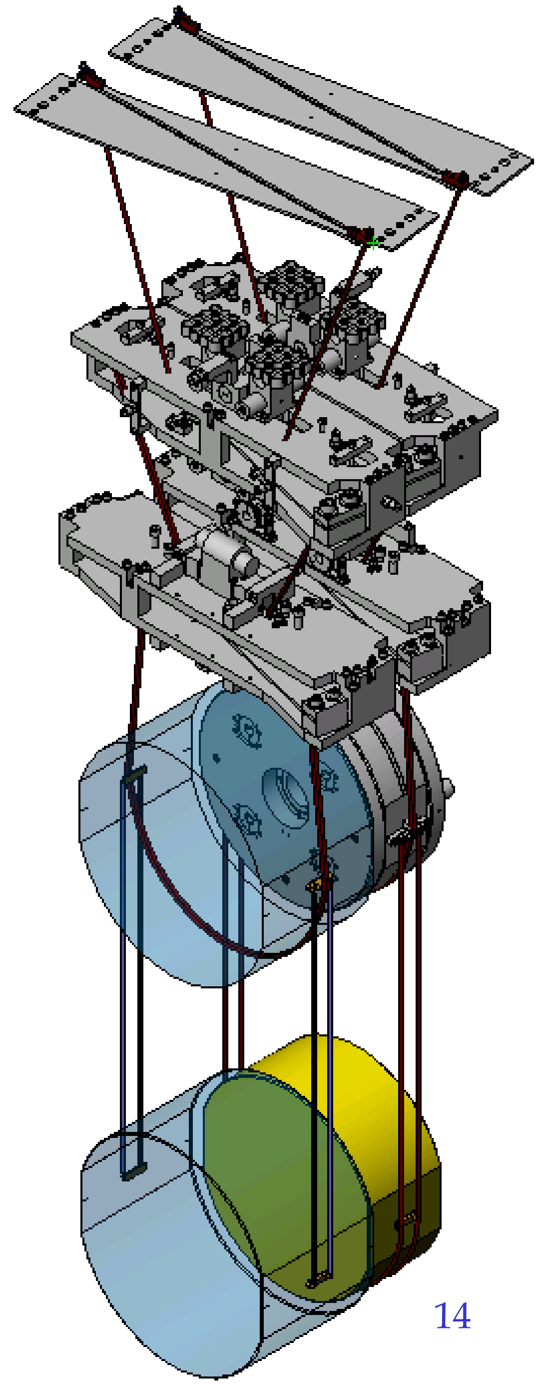
LIGO Observatory at Hanford, WA

Implementing one of Pirani's "particles"



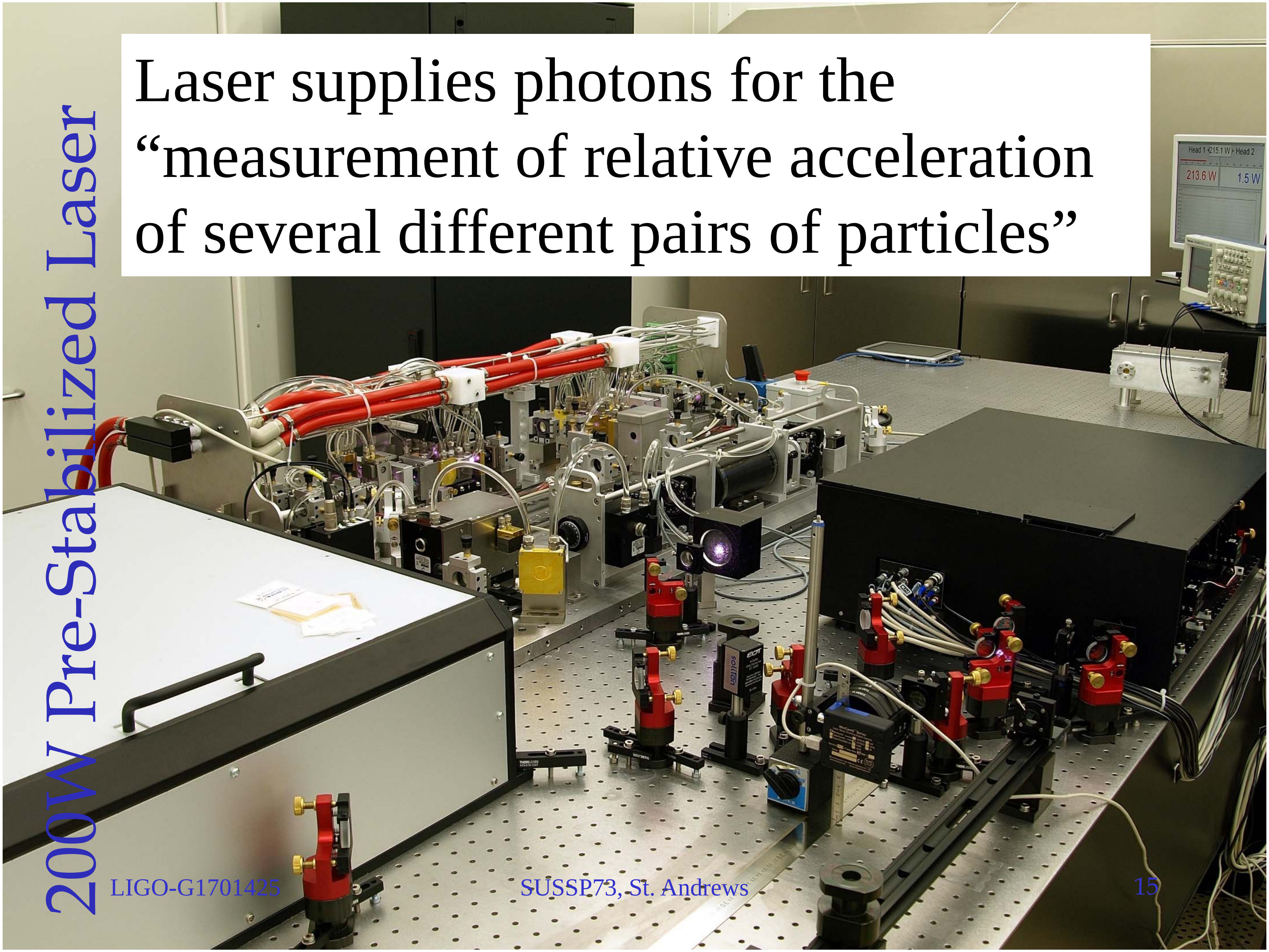
Making a “particle” that is quiet and free





200W Pre-Stabilized Laser

Laser supplies photons for the "measurement of relative acceleration of several different pairs of particles"



Gravitational wave detection is almost impossible

What is required for gravitational wave detection to succeed:

- interferometry with free masses,
- with strain sensitivity of 10^{-21} (or better!),
- (which is equivalent to ultra-subnuclear position sensitivity),
- in the presence of much larger noise.

Interferometry with free masses

What's "impossible": everything!

Mirrors need to be very accurately aligned (so that beams overlap and interfere) and held very close to an operating point (so that output is a linear function of input.)

Otherwise, interferometer is dead or swinging through fringes.

Michelson bolted everything down.

Strain sensitivity of 10^{-21}

Why it is "impossible":

Sensitivity h_{rms} can be expressed as

$$h_{rms} \sim \frac{\text{precision to which we can compare arm lengths}}{\text{length of arms}}.$$

Natural "tick mark" on interferometric ruler is one wavelength.

Michelson could read a fringe to $\lambda/20$, yielding h_{rms} of a few times 10^{-9}.

Ultra-subnuclear position sensitivity

Why people thought it was impossible:

- Mirrors made of atoms, 10^{-10} m.
- Mirror surfaces rough on atomic scale.
- Atoms jitter by large amounts.

Large mechanical noise

How large?

Seismic: $x_{rms} \sim 1\ \mu$m.

Can you filter it enough?

Thermal:

- mirror's CM: ~ 3 x 10^{-12} m.
- mirror's surface: ~ 3 x 10^{-16} m.

No filtering is possible. Can lower the temperature, but by enough?

Gravitational wave detection does work!

All of these challenges sound impossible.

And yet, all of them can be met.

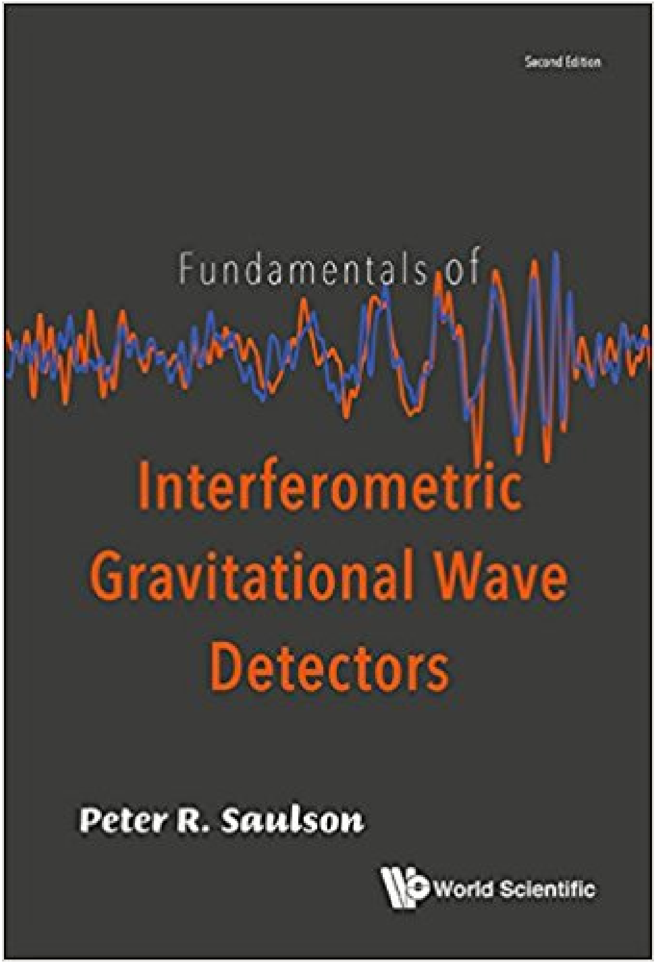
Detectors have now seen signals whose peak strain is 10^{-21} with signal-to-noise ratios of more than 20.

But how is it possible?

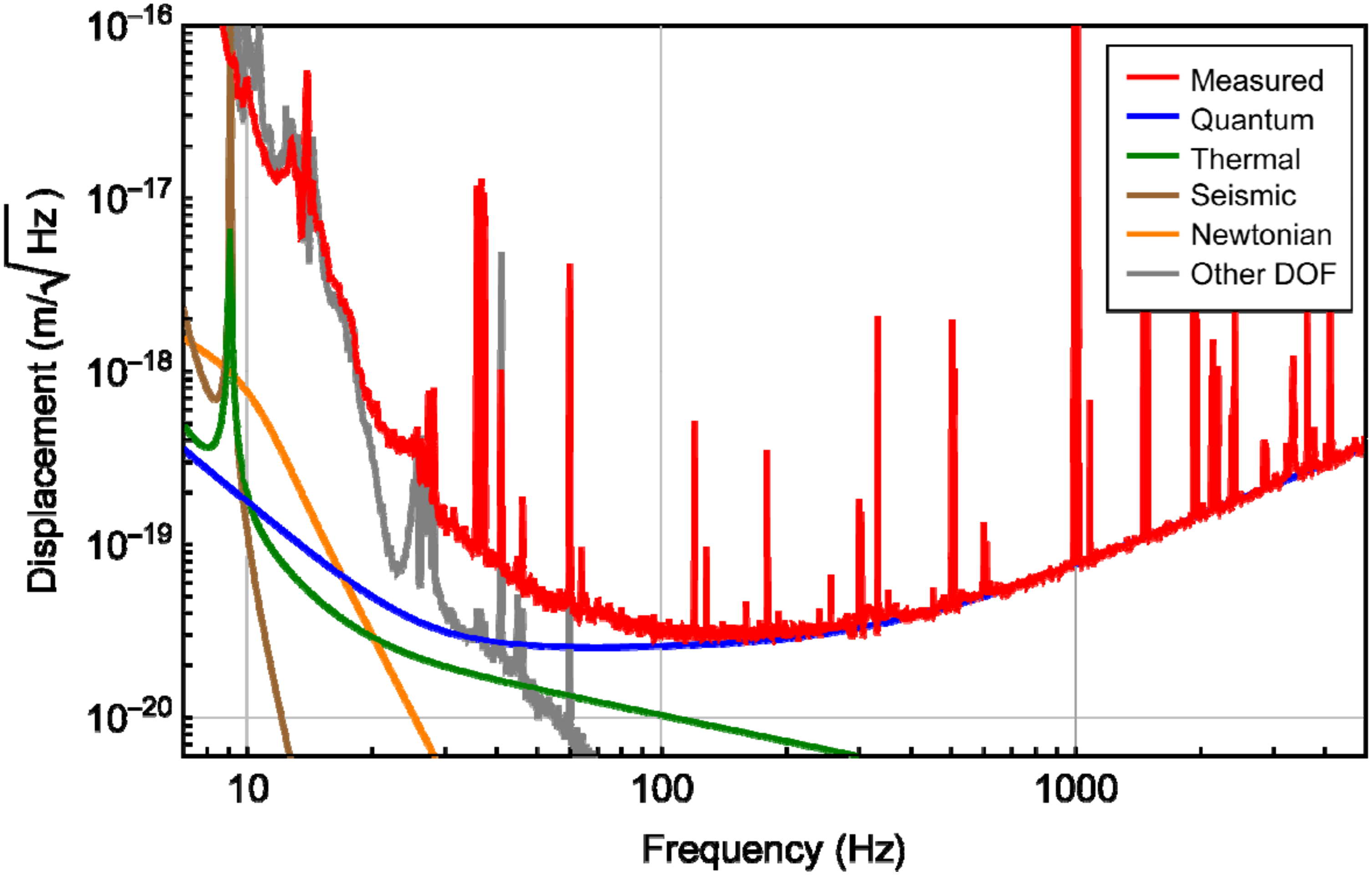
If you really want to learn the basics of how interferometers work …

… read the (2nd edition of) the book that I wrote for beginners.

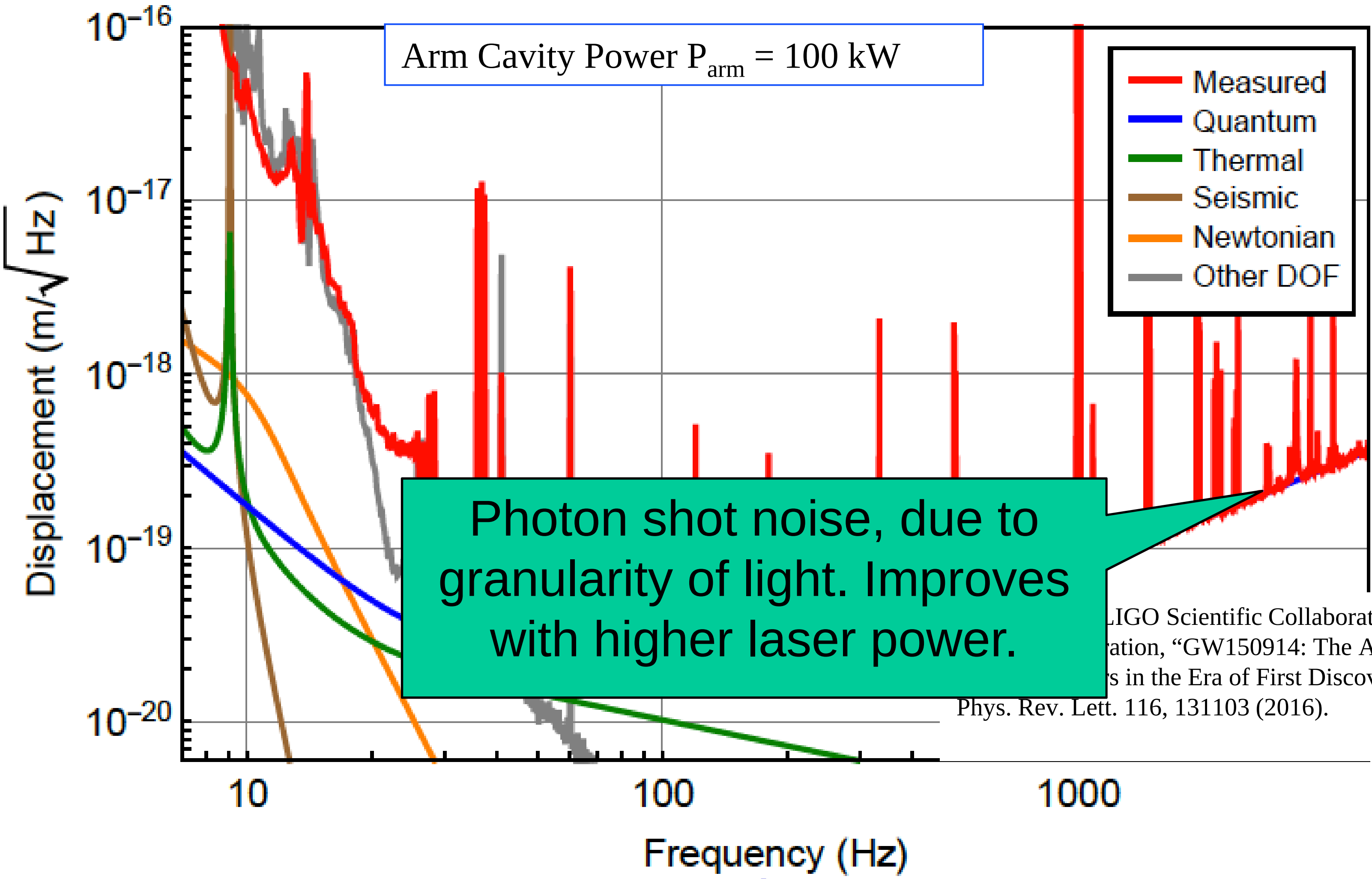
http://www.worldscientific.com/worldscibooks/10.1142/10116



Noise spectrum of aLIGO at the time GW150914 was found



Advanced LIGO detector sensitivity in 2015



…LIGO Scientific Collaboration and …ation, "GW150914: The Advanced …s in the Era of First Discoveries", Phys. Rev. Lett. 116, 131103 (2016).

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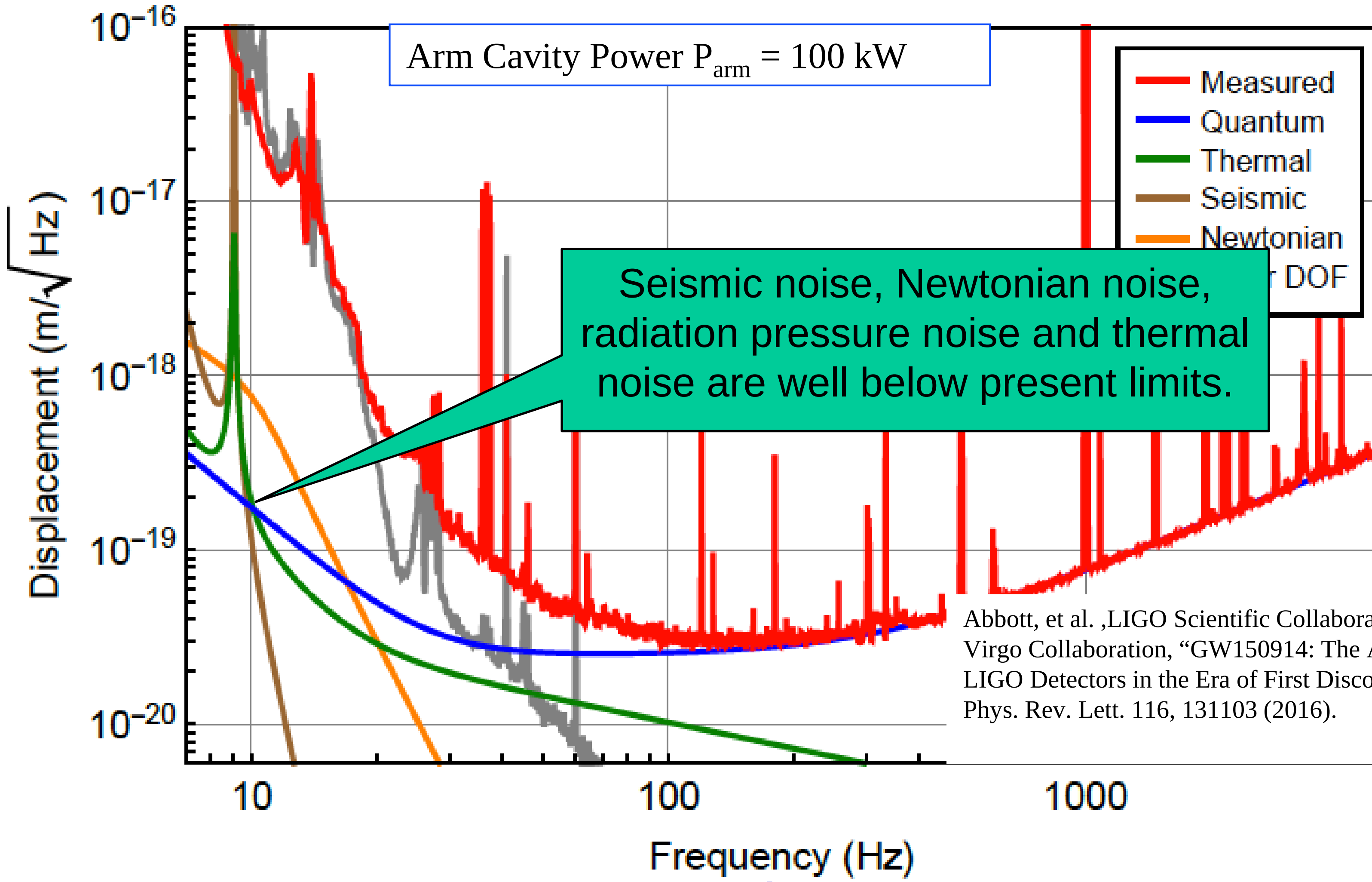

Abbott, et al. ,LIGO Scientific Collaboration and Virgo Collaboration, “GW150914: The Advanced LIGO Detectors in the Era of First Discoveries”, Phys. Rev. Lett. 116, 131103 (2016).

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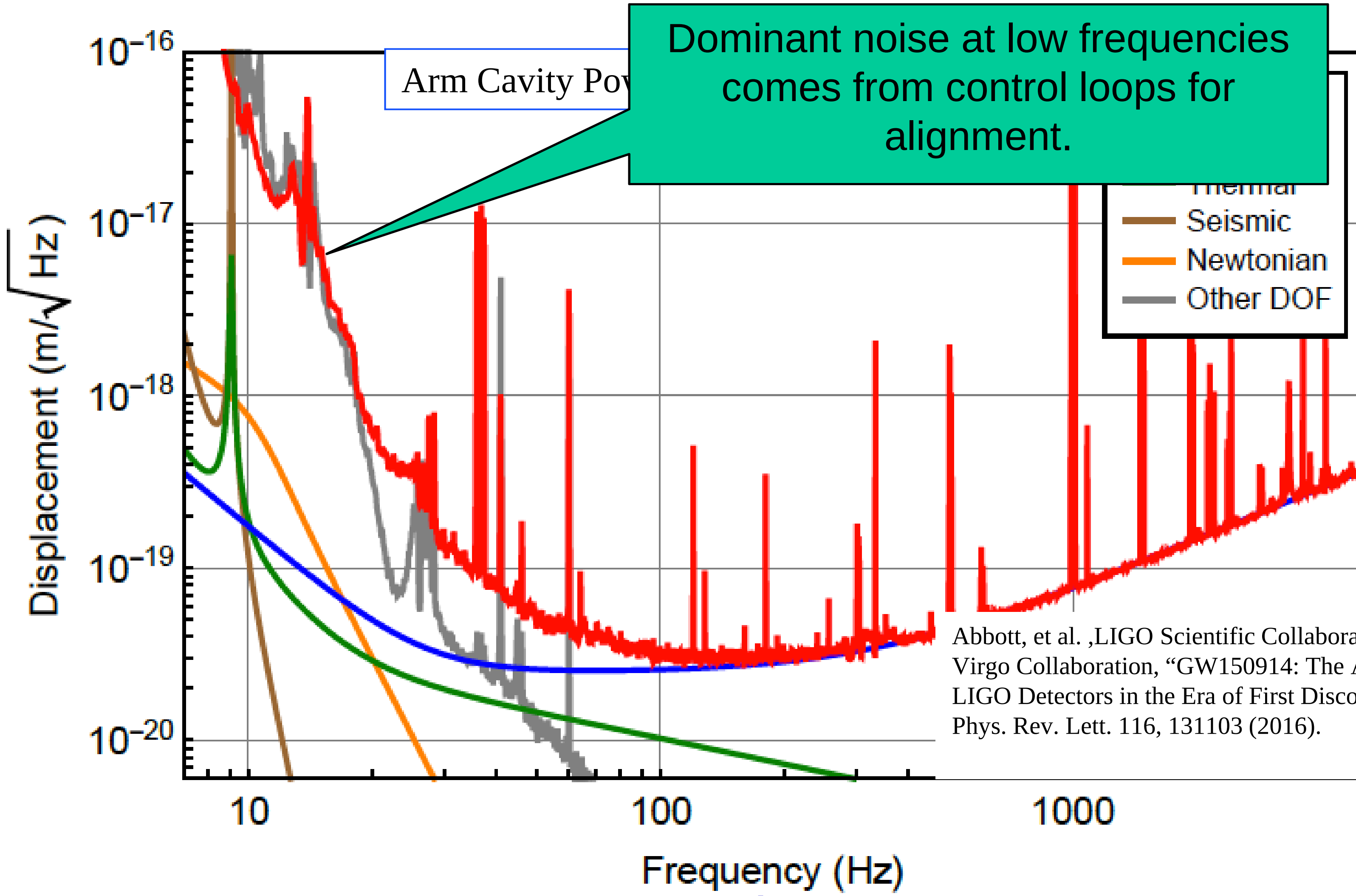

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The Fourier transform

The Fourier transform *X(f)* of *x(t)* is defined as

$$X(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

This measures the amount of a sine and cosine of each frequency *f* that it takes to build up the function *x(t)*.

Defines the relation between "the time domain" and "the frequency domain."

The power spectrum

One way to define the *power spectrum* $S_x(f)$ is by

$$S_x(f) = |X(f)|^2 / T.$$

Like the Fourier transform, it measures the admixture of sinusoids of all frequencies *f* that make up the time series *x(t)*;
however, it throws away the phase information (sines vs. cosines.)

Interpretation of the power spectrum

Conceptual way of measuring the power spectrum:

Apply signal *x(t)* to a bank of bandpass filters, each with 1 Hz pass band width, band centers at each integer frequency.

Compute the mean-square value of the output of each filter, and display as a function of *f*.

N.B.: If you sum up all outputs of all filters, then you recover the mean-square value of *x(t)*. Thus, the units of the power spectrum must be [units of *x*]^2/Hz.

The amplitude spectral density

Experimenters have limited minds, and find it easier to get their minds around something that doesn't square the units of *x(t)*. So we often use the amplitude spectral density

$$x(f) \equiv \sqrt{x^2(f)}$$

Its units are [units of *x(t)*]/$Hz^{1/2}$.

Why /$Hz^{1/2}$? Each frequency "bin" of the spectrum of a random time series is independent of the others. So they add in quadrature.

LIGO's sensitivity goal

Earlier, I loosely gave Advanced LIGO's sensitivity as $h \sim 10^{-22}$. What did I mean?

We want the standard deviation of strain measurements averaged over the 10 msec duration of, say, a signal from a supernova or black hole ringdown to be 10^{-22}.

Let's convert this spec to power spectrum language

$$\int_{50\text{ Hz}}^{150\text{ Hz}} h^2(f)\,df = \left(10^{-22}\right)^2.$$

This means we want a noise amplitude spectral density near 100 Hz of

$$h(f = 100\text{ Hz}) = 10^{-23} / \sqrt{\text{Hz}}.$$

Displacement noise goal

What spectrum of displacement noise is consistent with this goal?

We'll have four key mirrors (two in each arm to make the Fabry-Perot cavities, see later lecture.)

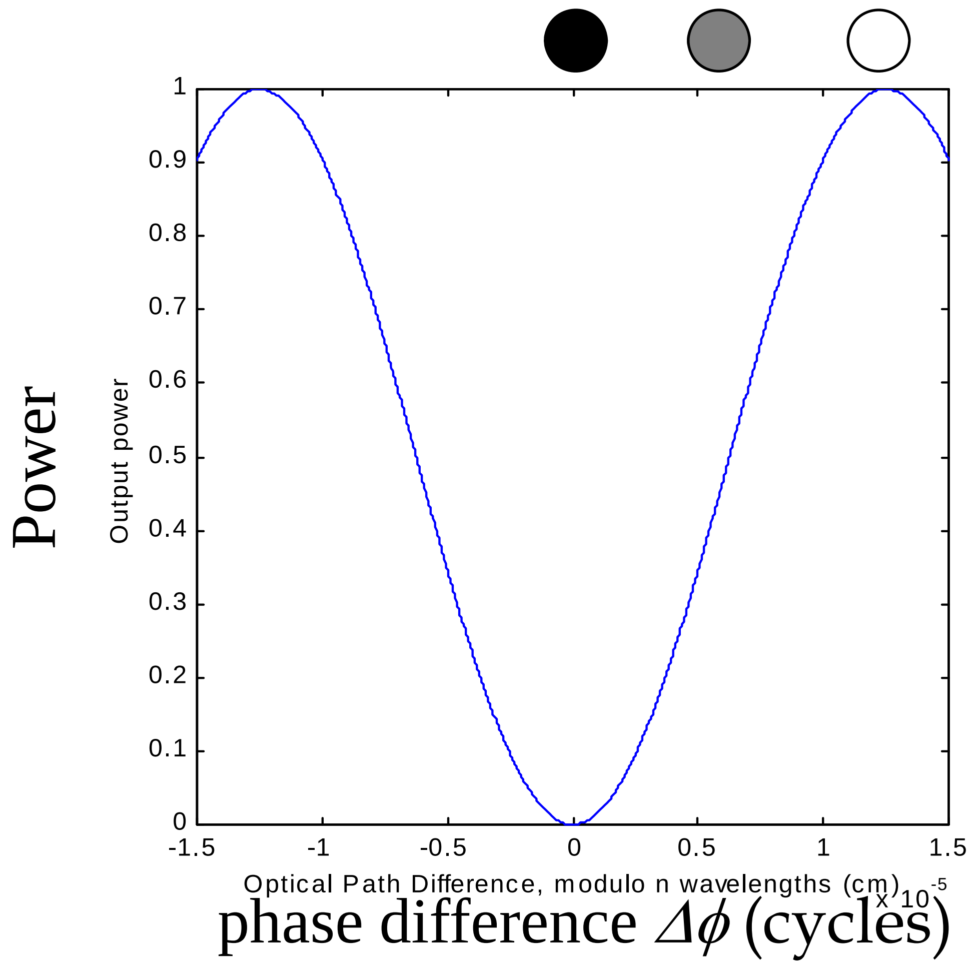
If their displacement noises are incoherent, they add in quadrature. The net result is

$$h_{disp}(f) = \frac{2}{L} x(f).$$

Thus, we need $x(f) = 2\ 10^{-20}$ m/Hz$^{1/2}$.

Shot noise

Interferometer output vs. arm length difference



Sensitivity of interferometer with "on/off" readout

If we only distinguish between bright and dark output, interferometer wouldn't be very sensitive.

$$\Delta h_{crude} \sim \frac{\lambda / 2}{L_{optical}}.$$

$$\lambda \approx 1\,\mu m,$$

$$(L = 4\,km) \times (\mathrm{N} = 100)$$

$$\Delta h_{crude} \sim 10^{-12}$$

The "fringe-splitting" solution

We require ~10 more orders of magnitude in sensitivity, if we hope to see gravitational waves.

If so, then we need to know much more than whether we are on the bright or dark point of a fringe.

We need to know, to 1 part in 10^{10}, where we are in the fringe.

Is this possible? Yes.

Working out some rough numbers

We need to do 10^{10} times better than “on/off” measurement.

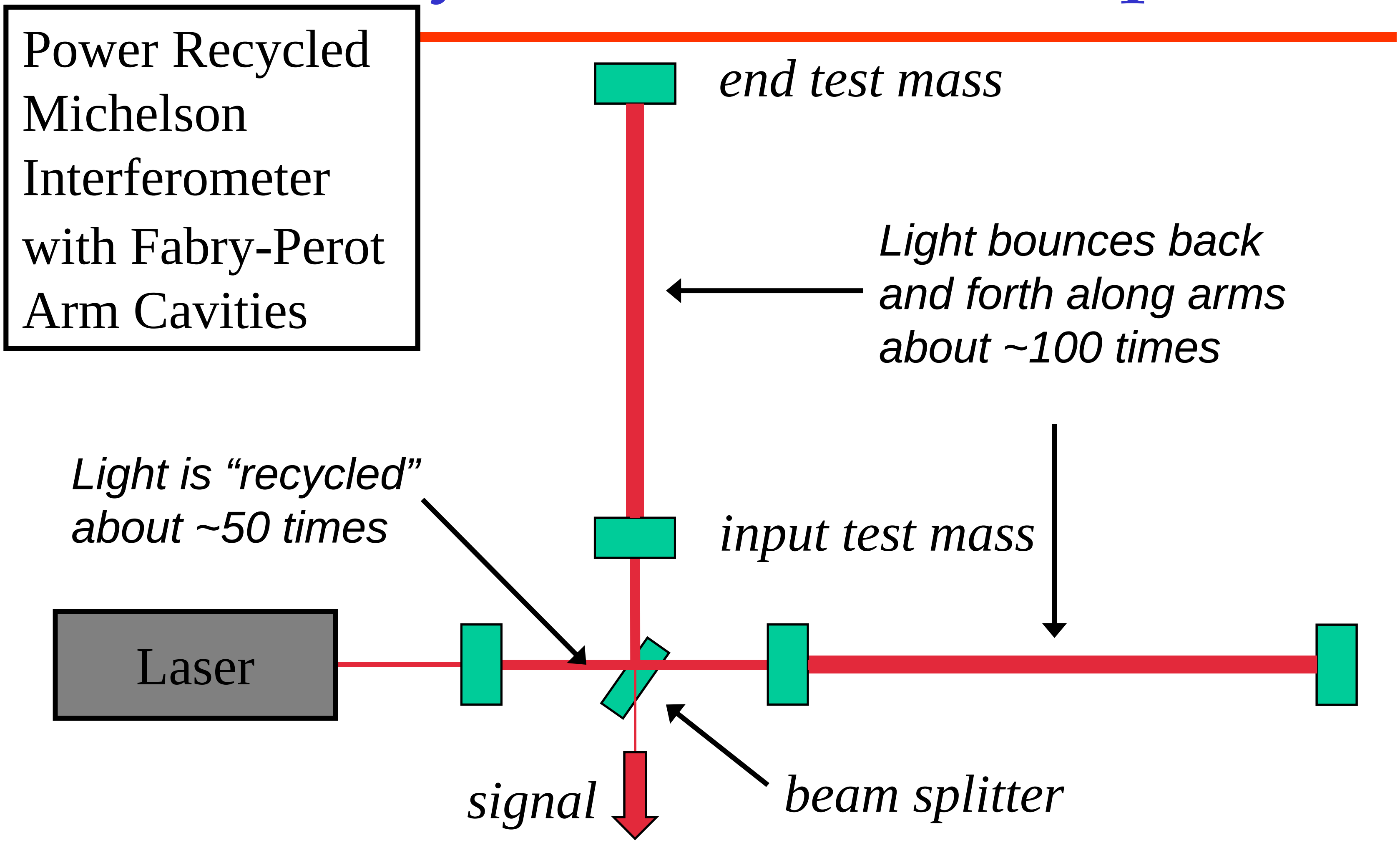
Thus, we require 10^{20} photons in each 0.01 sec measuring interval.

$$P_{in} = \frac{2\pi\hbar c}{\lambda}\overline{N}$$

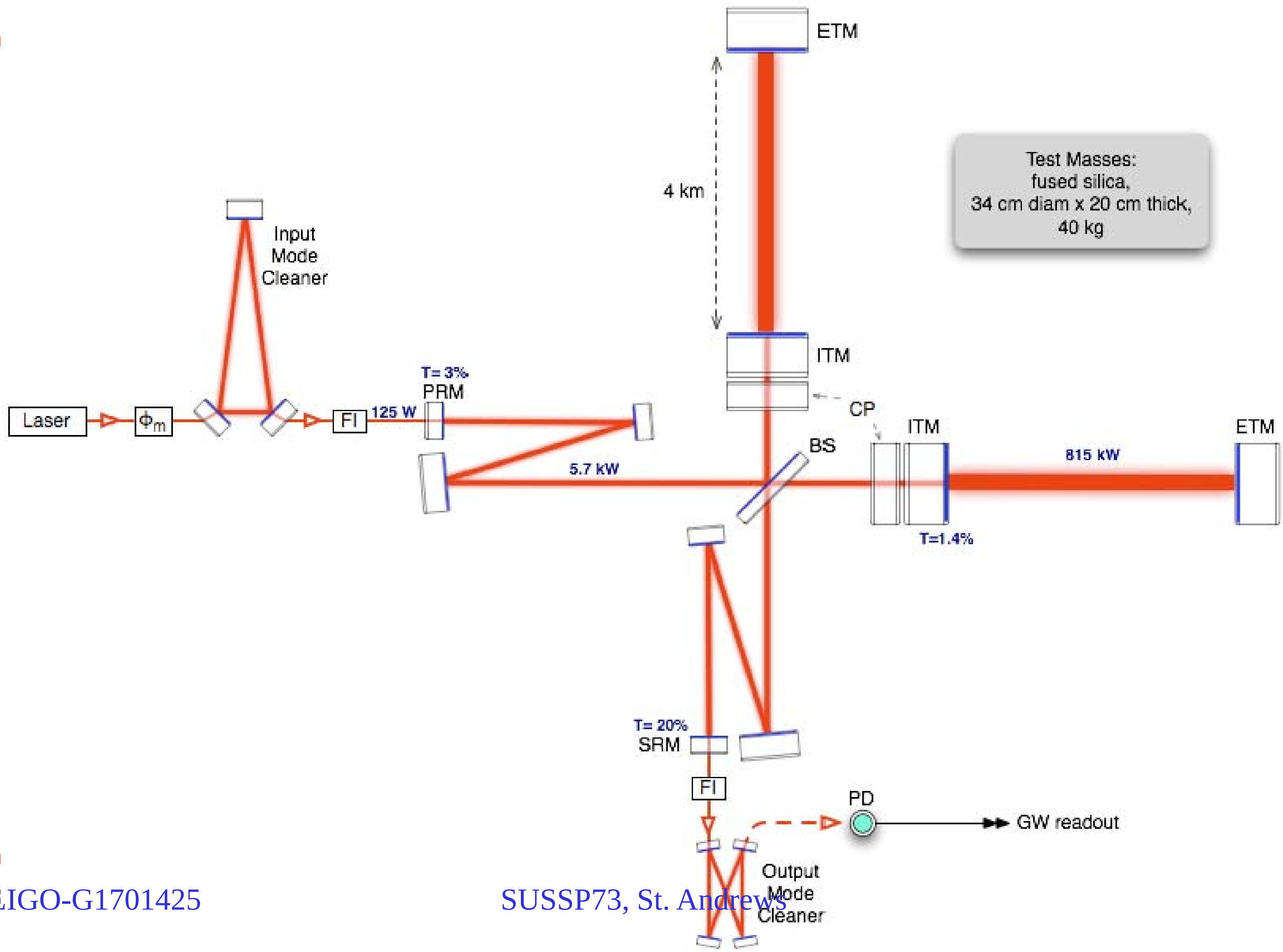
We need about 2 kW of input power.

We’ll did this with about 20 W of laser power, and the trick called “power recycling”.

To achieve the required sensitivity, need cleverer optics



The design has many refinements



Seismic noise

How strong is seismic noise?

Amplitude spectrum of seismic noise above 10 Hz is typically
10^{-9} m/Hz$^{1/2}$*(10 Hz/f)^2.

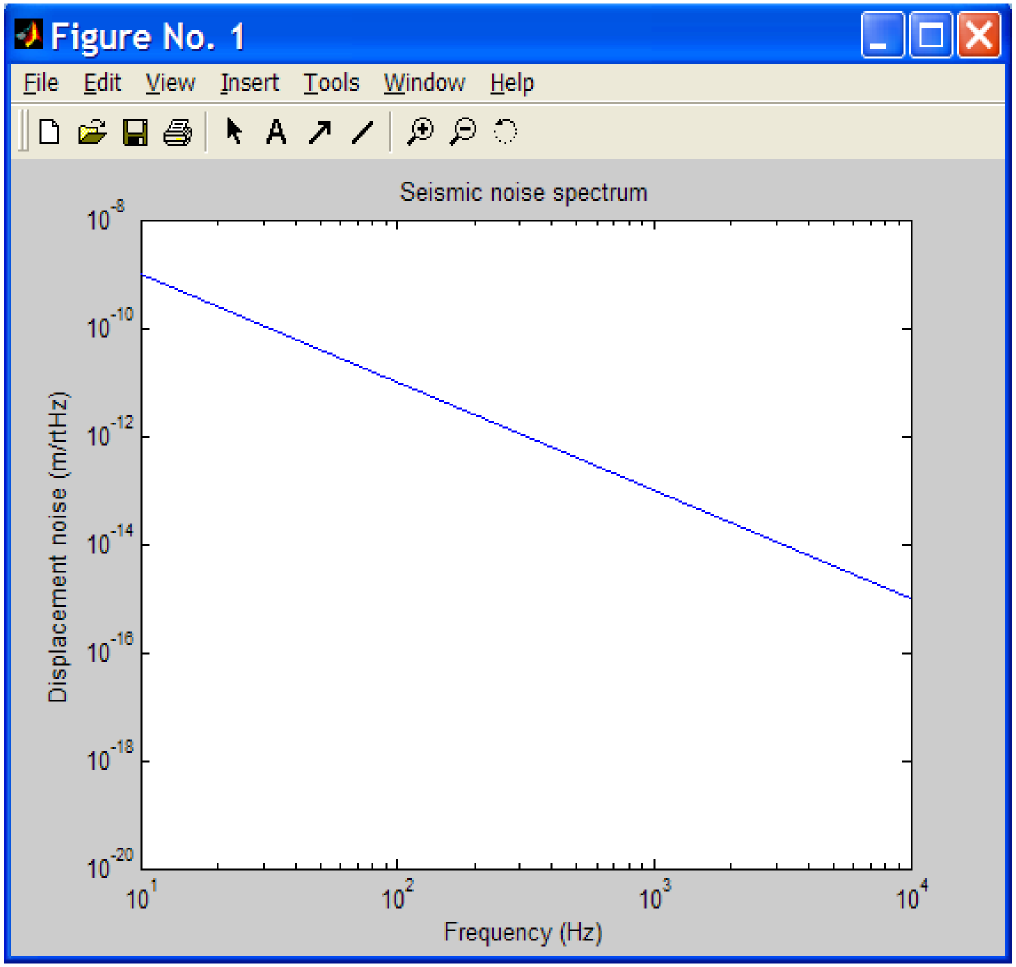
At a target frequency of 100 Hz,
$x(f) = 10^{-11}$ m/Hz$^{1/2}$,

far from $x(f) = 2\ 10^{-20}$ m/Hz$^{1/2}$.

Seismic noise is serious!

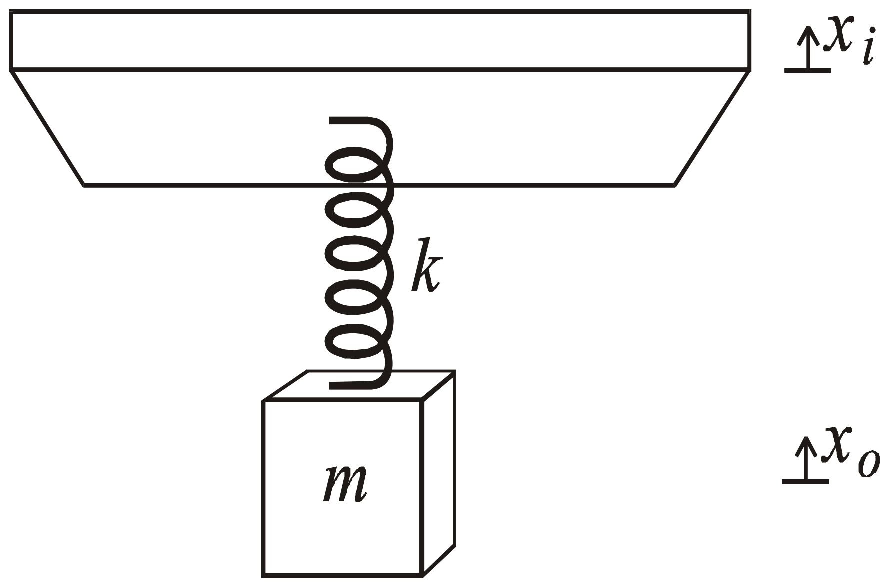
We need 9 orders of magnitude of isolation at 100 Hz.

Seismic noise spectrum



A simple harmonic oscillator as a vibration isolator

My favorite linear system, the simple harmonic oscillator, a.k.a. a mass on a spring.



The input is the position of the top of the spring, and the output is the position of the mass.

Equation of motion

The equation of motion of our canonical linear system is

$$m\ddot{x}_o + k\left(x_o - x_i\right) + b\dot{x}_o = 0.$$

The input is applied by moving the top of the spring, thus stretching the spring (the mass has inertia), so a Hooke's Law force is applied to the mass.

Finding the frequency response of a linear system

Derivation of frequency responses also involves easier math than finding the impulse response. Here's how.

Consider a sinusoidal input of frequency *f*:

$$x_i(t) = X_i(f)e^{i2\pi ft}.$$

Then, the output will also have a sinusoidal form, since the e.o.m. is linear.

$$x_o(t) = X_o(f)e^{i2\pi ft}.$$

Frequency response example (II)

Recall:

$$\frac{d}{dt} e^{i2\pi ft} = i2\pi f e^{i2\pi ft}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{i2\pi ft} = -(2\pi f)^2 e^{i2\pi ft}$$

Plug our *ansatz* into the e.o.m.

$$m\ddot{x}_o + k\left(x_o - x_i\right) + b\dot{x}_o = 0.$$

divide through by $e^{i2\pi ft}$ everywhere, and find

$$-m(2\pi f)^2 X_o + k(X_o - X_i) + i2\pi f b X_o = 0.$$

Finally, solve for $G(f) = X_o(f)/X_i(f)$

$$G(f) = \frac{k}{k + i2\pi f b - m(2\pi f)^2}.$$

Frequency response example (II)

$$G(f) = \frac{k}{k + i2\pi fb - m(2\pi f)^2}.$$

Q: Why are we happy to have done this?

A:

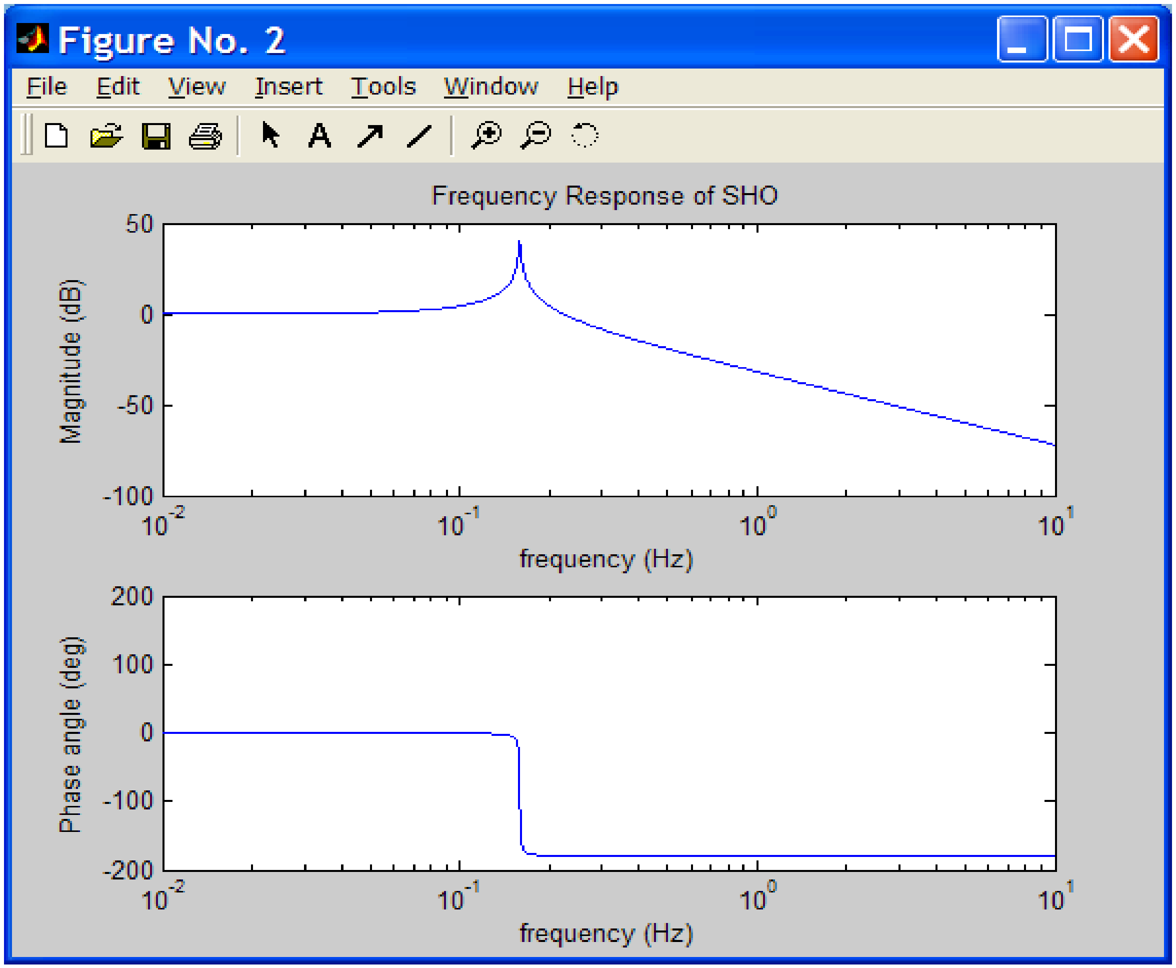
1. Using only simple algebra, we've solved a differential equation.
2. We can gain insight in the frequency domain that is hard to obtain in the time domain.

Bode plots

A frequency response is typically graphed in the form of a Bode plot (actually two graphs on the same logarithmic frequency scale.)

a) The magnitude of the frequency response is plotted on a logarithmic scale. The traditional units are deciBels (dB), given by Mag(dB) = $20 \log_{10} |G(f)|$.
b) The phase of $G(f)$ is plotted on a linear scale between –180 and +180 degrees.

Bode plot of our example's frequency response



Reading a Bode plot

The resonant frequency stands out as the place where response is largest. It isn't infinite, because of damping.

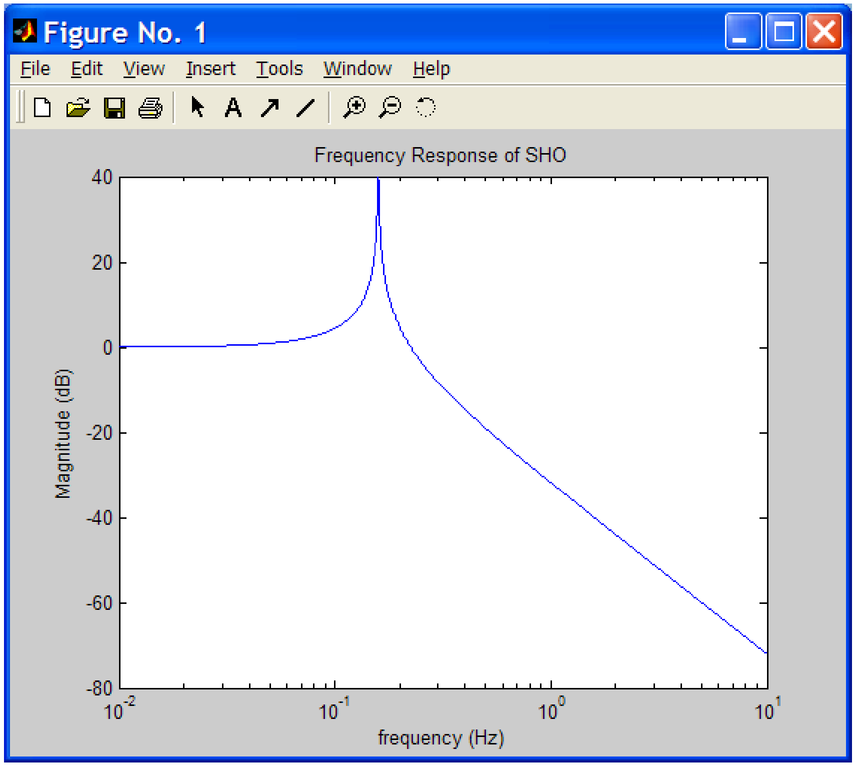
At $f << f_{res}$, response is unity (= 0 dB.) The mass tracks the motion of the top end of the spring. The dynamics is "stiffness controlled."

At $f >> f_{res}$, the mass moves less at higher frequencies (proportional to $1/f^2$, or –40 dB per decade), due to the inertia of the mass.

SHO as filter

A mass on a spring makes a good isolator.

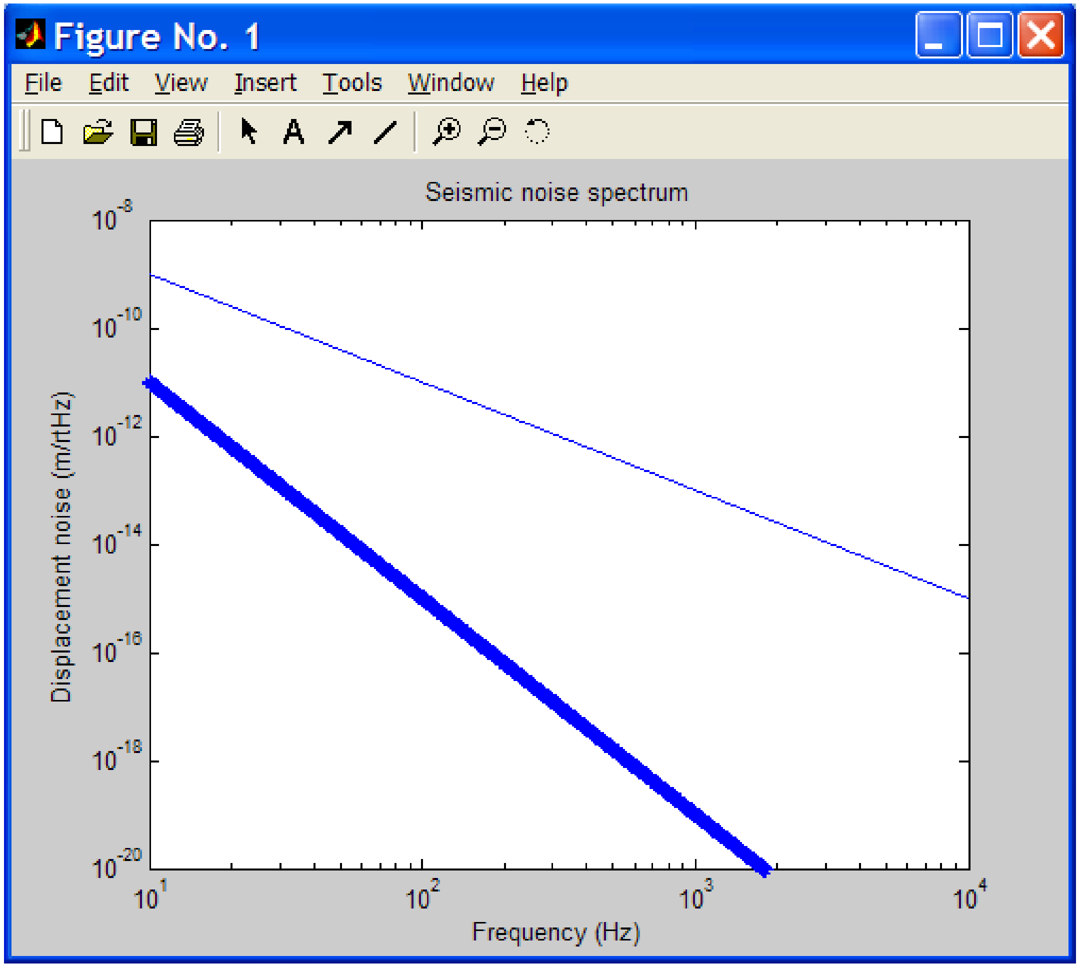
Frequency response goes like $1/f^2$ above resonance.



Pendulum as isolator

- One such SHO is built into our plans already – each test mass must be suspended as a pendulum, to allow it to respond freely to the gravitational wave.
  - It has a resonant frequency near 1 Hz.
  - Thus, we should multiply the input spectrum by $(1\ \text{Hz}/f)^2$ to find output spectrum (i.e., motion of mirror.)

Pendulum helps, but only makes seismic noise good enough at $f > 1$ kHz



Multiple stages of isolation for better filtering

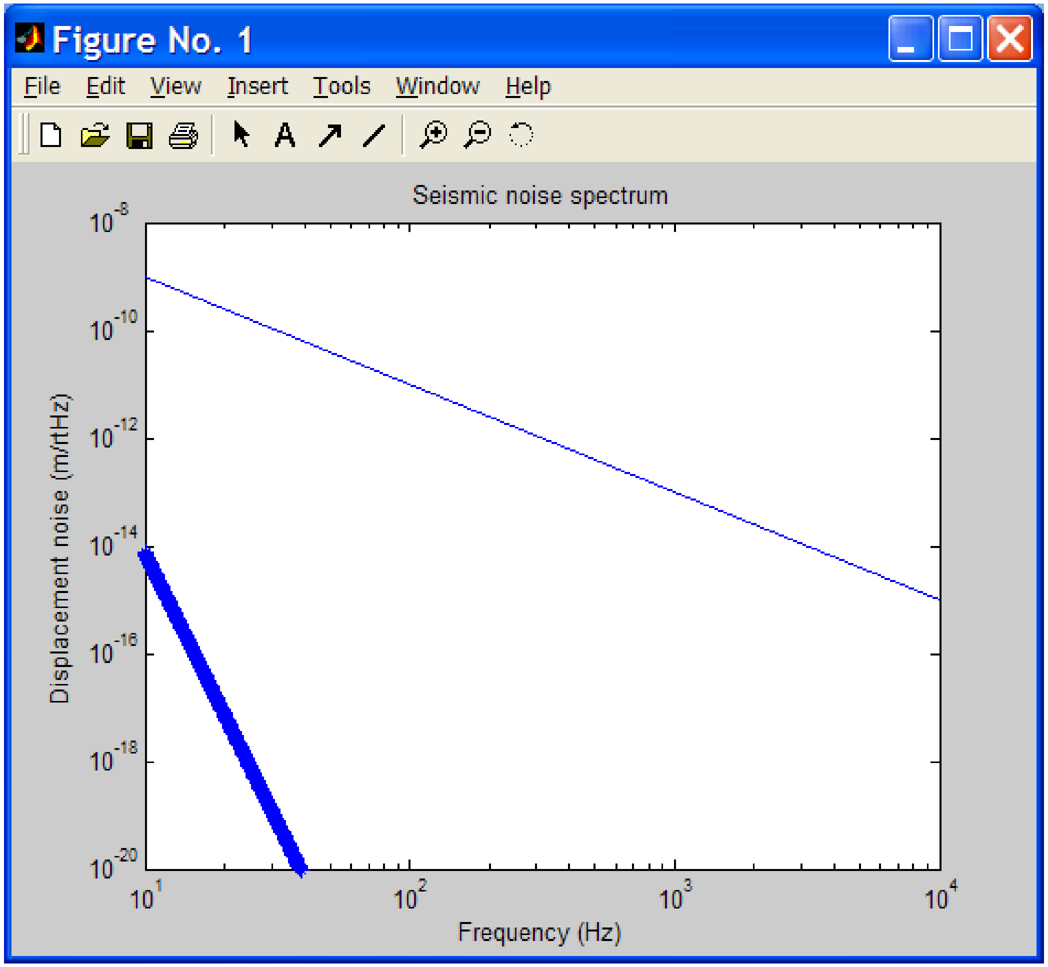
One SHO makes a good filter, but not good enough.

If we make a chain of N oscillators, we have a coupled system with N resonances, above which the frequency response is

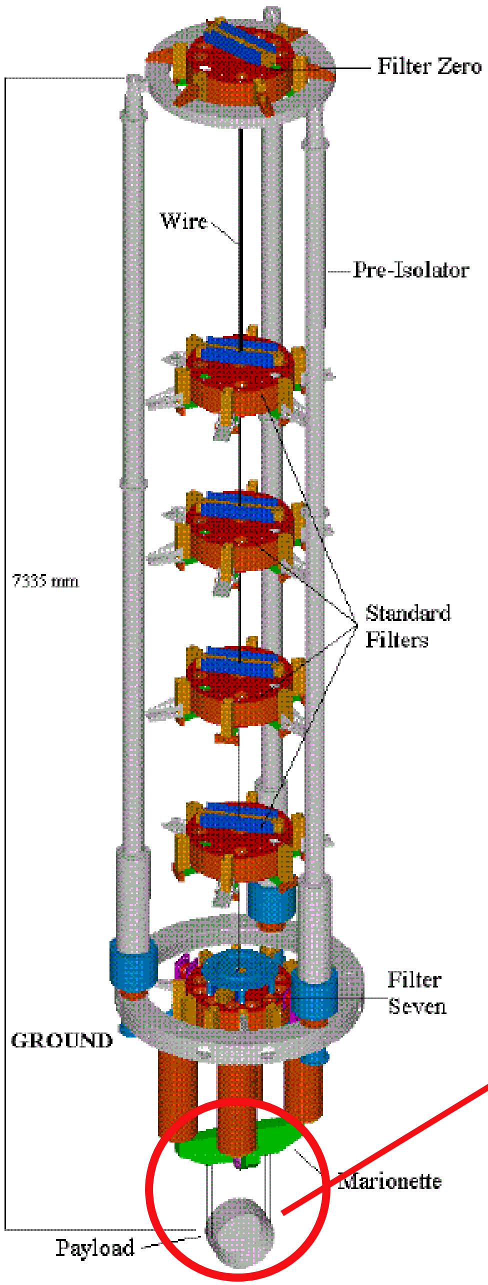
$$G(f) = \left(\frac{f_0}{f} \right)^{2N}.$$

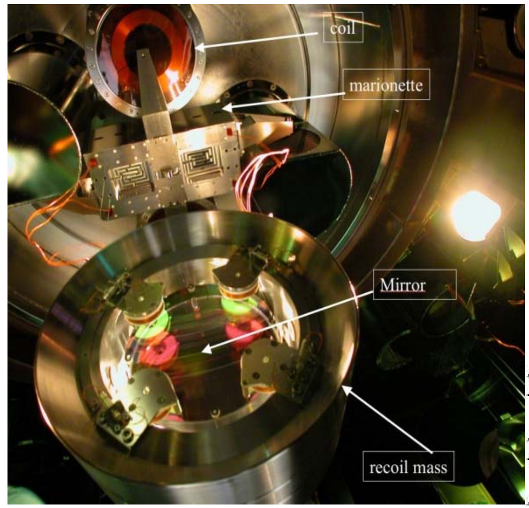
Just need to build enough stages of isolation.

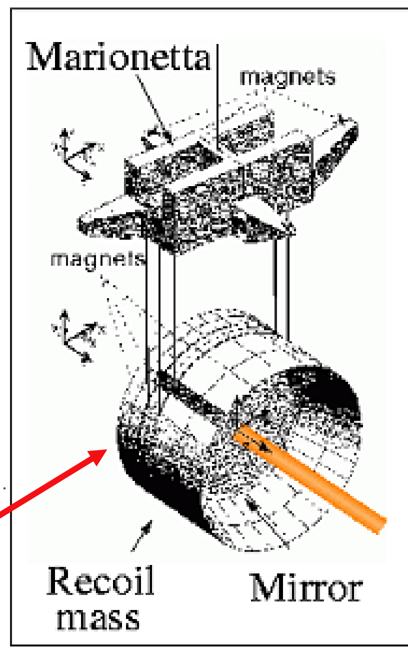
Here’s what 3 stages would do

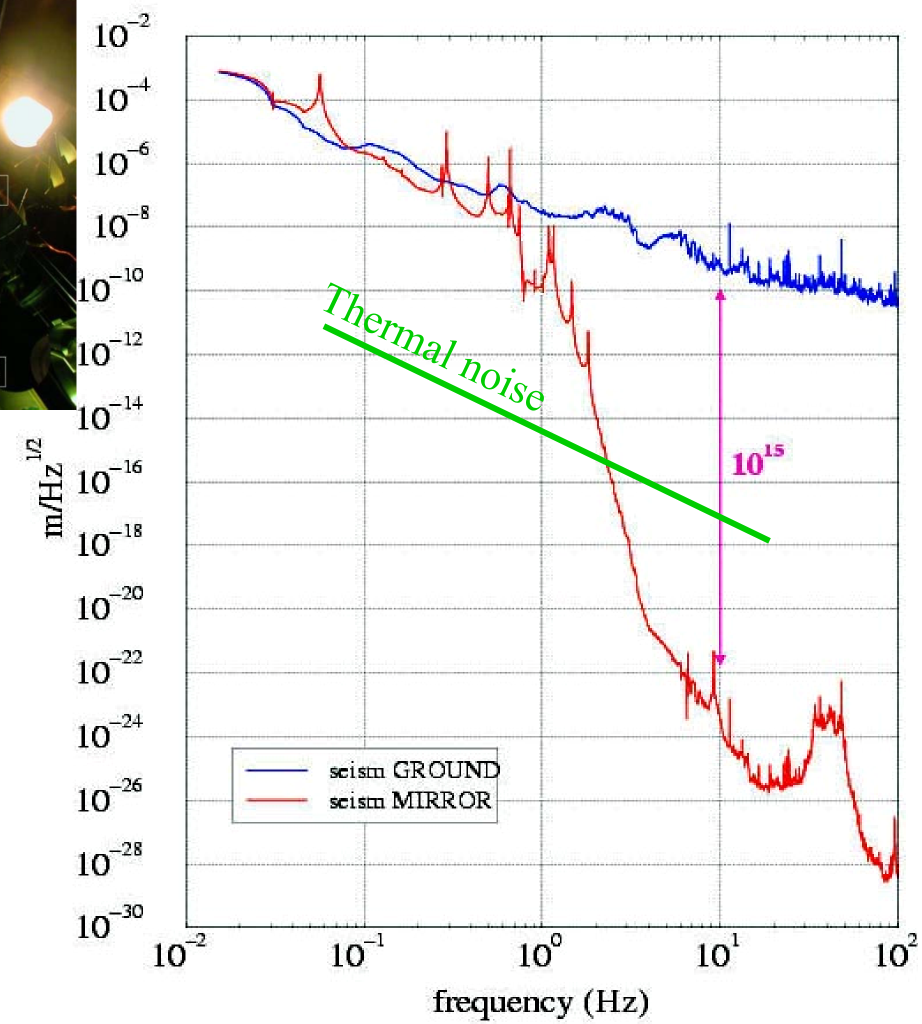


Virgo's Free falling test masses

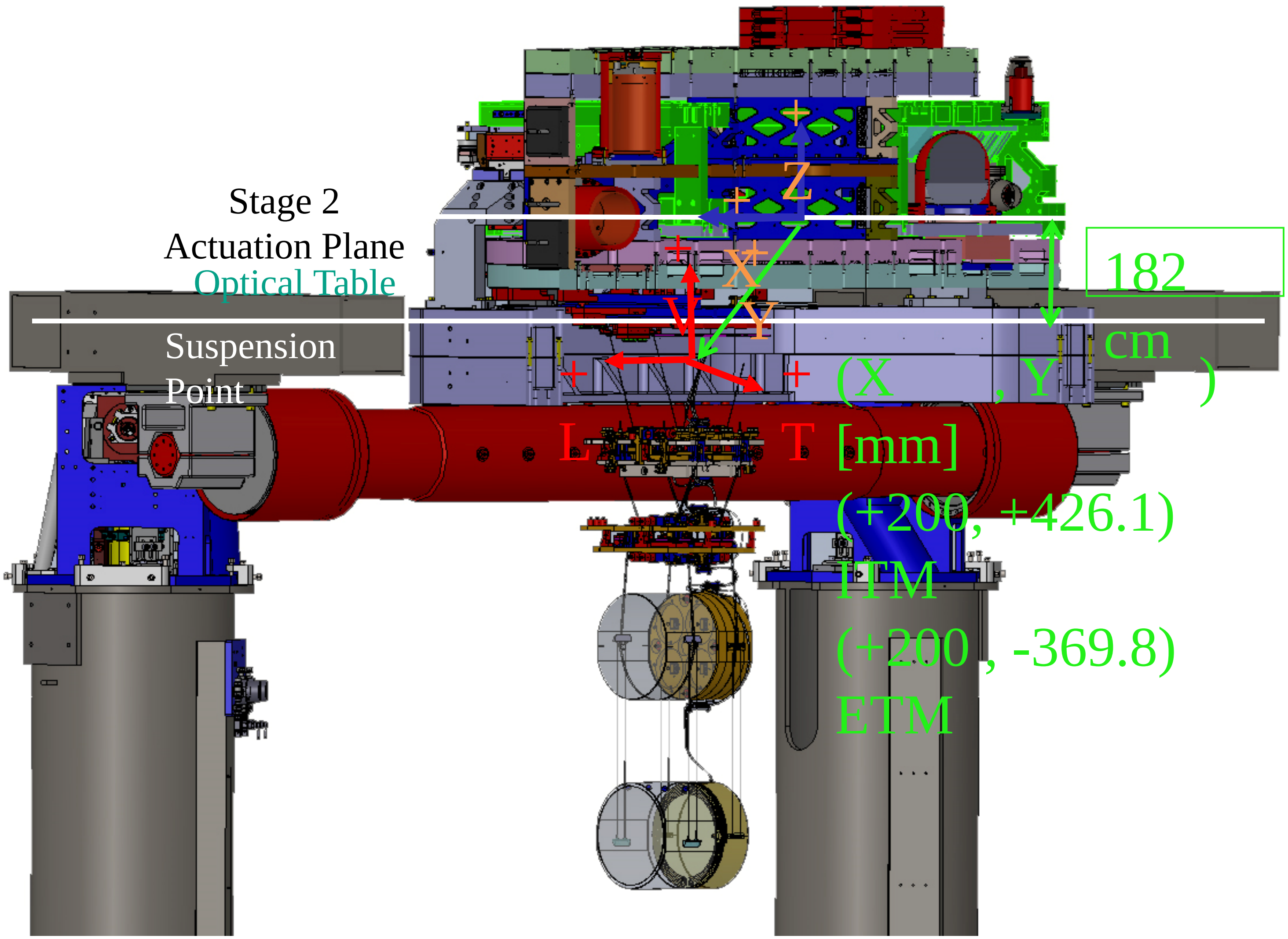








aLIGO also has active isolation



Thermal noise

Large mechanical noise

How large?

Seismic: $x_{rms} \sim 1\ \mu$m.

Brownian motion:

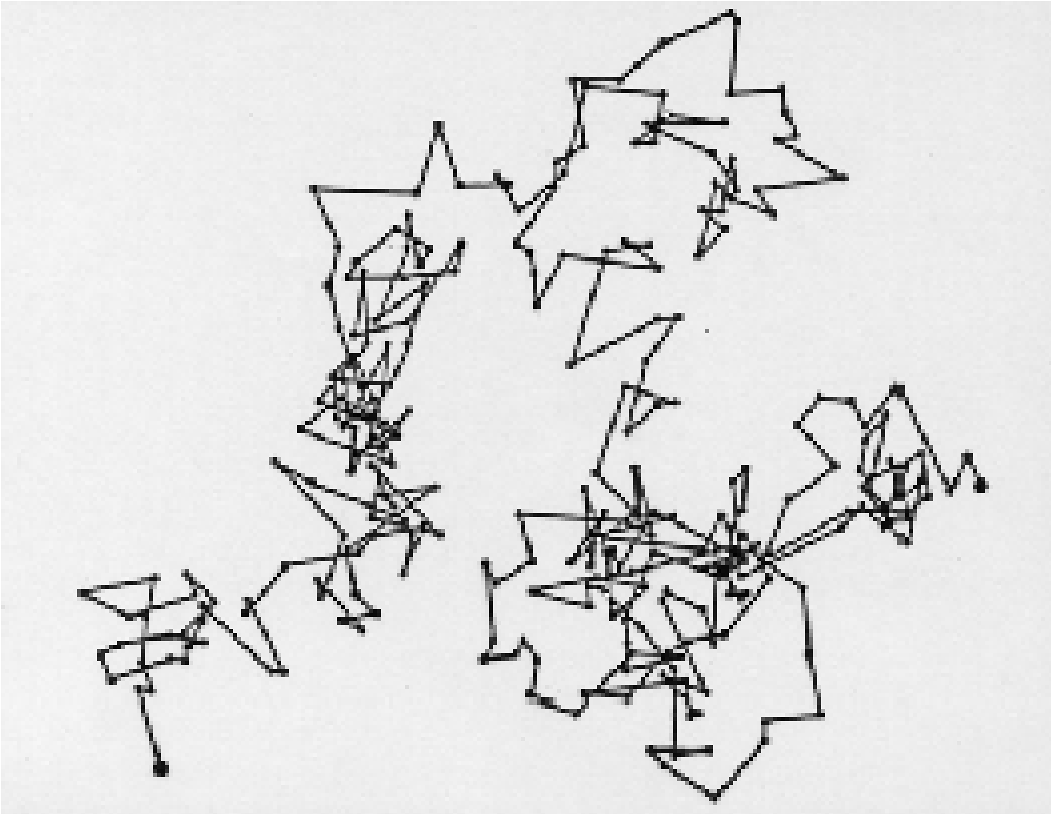
- mirror's CM: ~ 3 x 10^{-12} m.
- mirror's surface: ~ 3 x 10^{-16} m.

If so, can we detect gravitational waves?

Firstly, we'll focus on Brownian motion.

Brownian motion

In 1827,
Robert Brown
noticed incessant
jiggling of tiny
particles
suspended in
water, as seen
through a
microscope.



Einstein's key contribution

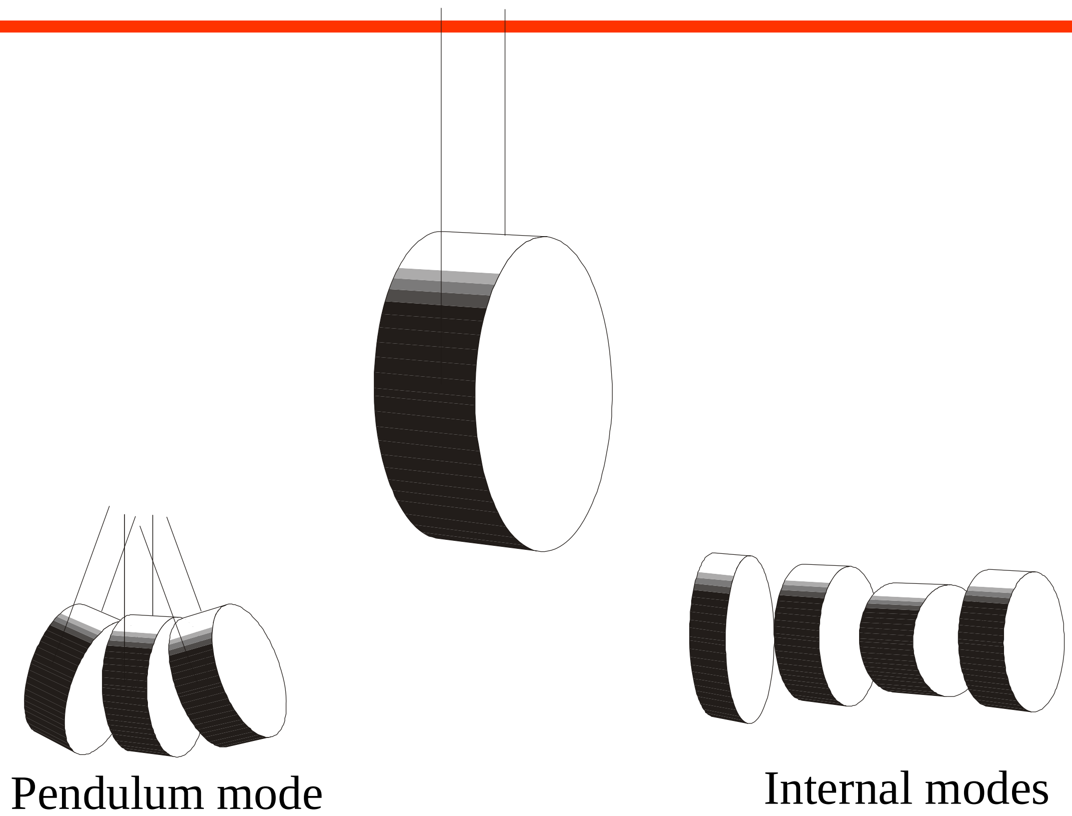
1905: Einstein shows that a Brownian particle's random walk obeys

$$\left\langle x^2(t) \right\rangle = 2k_B TBt$$

where B is a coefficient called the *mobility* of the particle (which depends on friction felt by the Brownian particle.

The first link clear and incontrovertible link between fluctuation and dissipation.

Interferometer suspensions



$$x^2(f) = \frac{2}{\pi}\frac{k_B T}{k}\frac{1}{f\left[\left(1 - f^2/f_0^2\right)^2 + \phi^2\right]}\phi(f)$$

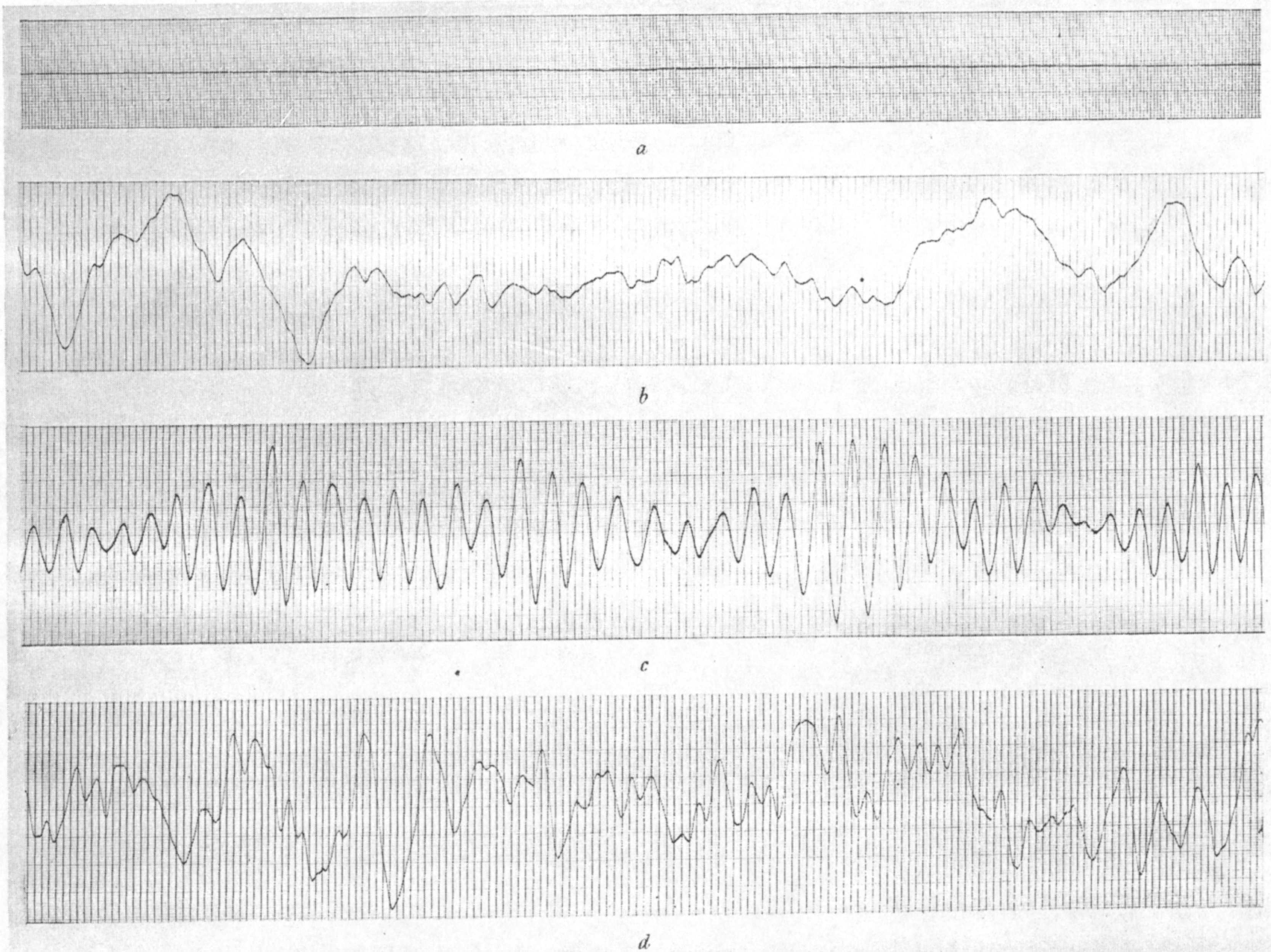


FIGURE 1. *a*. Record of secondary trace, with fixed primary mirror, at approximately twice the sensitivity used in records (*b*), (*c*) and (*d*). Vertical lines at 1 sec. intervals. *b*, *c*, *d*. Records of secondary trace with primary galvanometer (resistance 10·76Ω, free period 2·17 sec.) [illegible] circuited (*b*), on open circuit (*c*), and nearly critically damped with external resistance 133·24 ohms (*d*). Secondary galvanometer (free [illegible] Sensitivity about 3·2 × 10^{-12} amp. per large division. Vertical lines at 0·5 sec. intervals.

Same lesson in frequency domain