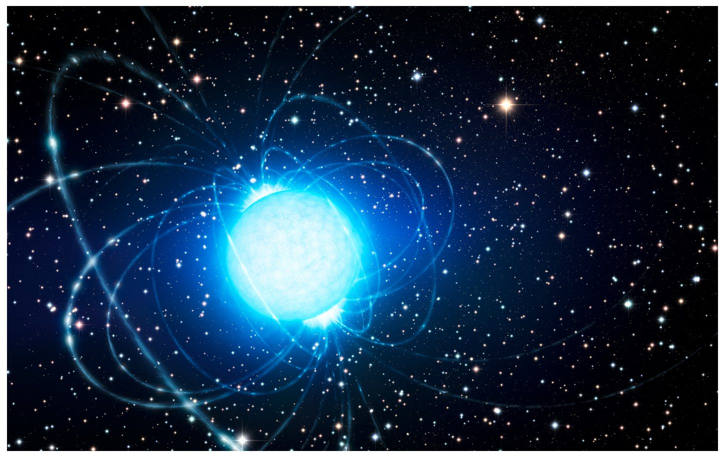
Lecture 2

Spinning out of control

St Andrews summer school 2017



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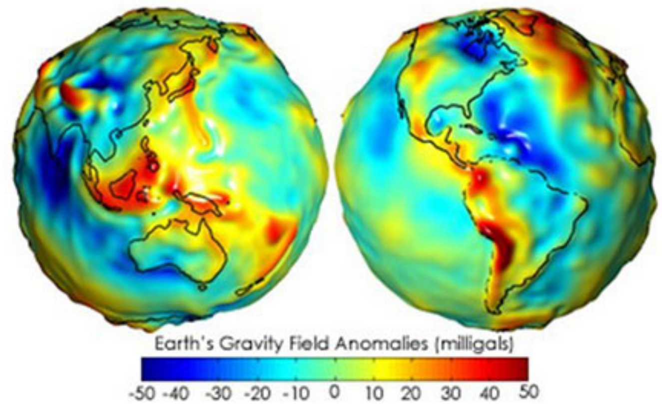
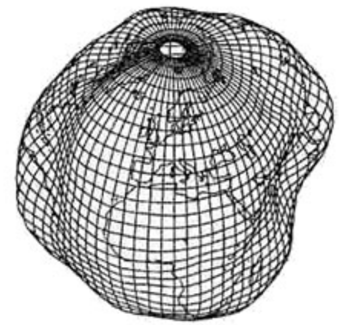
In principle, any rotating deformed body (like the Earth!) radiates GWs.





Estimate the GW emission from the quadrupole formula.

For a rotating rigid body, we have

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \ , \quad \text{or} \quad v^i = \epsilon^{ijk}\Omega_j x_k$$

where Ω^i is the angular frequency and x^i is the position vector. We have the kinetic energy

$$E = \frac{1}{2}\int \rho v^2 dV = \frac{1}{2}\int \rho(\boldsymbol{\Omega} \times \boldsymbol{r})^2 dV = \frac{1}{2}\int \rho[\Omega^2 r^2 - (\Omega^i x_i)^2]dV$$

$$= \frac{1}{2} I_{ij}\Omega^i\Omega^j$$

and the angular momentum

$$J^i = \epsilon^{ijk}\int \rho x_j v_k d^3x = \epsilon^{ijk}\epsilon_{klm}\int \rho x_j \Omega^l x^m dV$$

$$= \Omega^j \int \rho(\delta_{ij} r^2 - x_i x_j)dV = I_{ij}\Omega^j$$

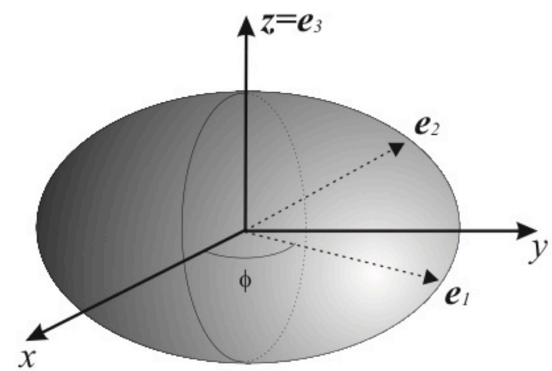
For the GWs we need the reduced quadrupole moment

$$I_{jk} = -\mathcal{I}_{jk} + \frac{2}{3}\delta_{jk} \int \rho r^2 d^3x$$

where the second term (the trace) is constant.

Note: In general easier to work in the body frame where I_{jk} is diagonal.

Taking the star to rotate around the z-axis, we have



$$I_{xx} = -I_{yy} = \frac{1}{2}(I_1 - I_2)\cos 2\varphi$$

$$I_{xy} = I_{yx} = \frac{1}{2}(I_1 - I_2)\sin 2\varphi$$

And

$$\frac{dE}{dt} = \frac{G}{5c^5}\left\langle \dddot{I}_{xx}^2 + 2\dddot{I}_{xy}^2 + \dddot{I}_{yy}^2 \right\rangle = \frac{32G}{5c^5}(I_1 - I_2)^2\Omega^6$$

Next we assume that the star is a rotating ellipsoid (semiaxes a_i) and introduce the ellipticity (for small deformations):

$$I_1 - I_2 = \frac{2\epsilon M(a_1 + a_2)^2}{5} \approx \frac{2\epsilon M R^2}{5} = \epsilon I_0$$

In the end we arrive at

$$\frac{dE}{dt} \approx \frac{32G}{5c^5}\epsilon^2 I_0^2 \Omega^6$$

where $I_0 = 2MR^2/5$ is the moment of inertia for a uniform density sphere.

Compare the result to observed pulsar spindown:

$$\frac{\dot{P}}{P} = -\frac{\dot{E}}{2E} = -\frac{32G}{5c^5}\epsilon^2 I_0 \Omega^4$$

Example: In the case of the Crab pulsar, we have $P=33$ ms and

$$\dot{P}_{\rm obs} \approx 4.2 \times 10^{-13}$$

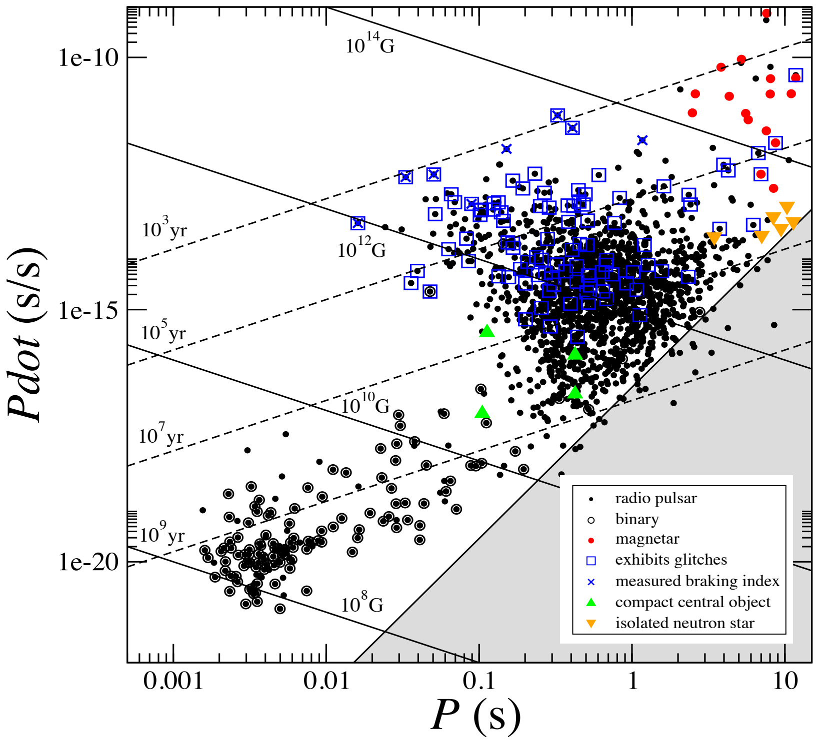
Assuming typical NS parameters (1.4 solar masses & 10 km) we have

$$\dot{P}_{\rm GW} \approx -8 \times 10^{-7}\epsilon^2$$

So... If the Crab pulsar spins down entirely due to GW emission, we need

$$\epsilon \approx 7 \times 10^{-4}$$

This is a “useful” result, but we know it is an upper limit. The observed braking index (second derivative) is $n=2.51$, closer to the canonical value of 3 for EM emission than the expected 5 for GWs.



In general, we have the spindown limit from observed systems;

$$\varepsilon = \left[\frac{5\dot{P}P^3}{32(2\pi)^4 I_0} \right]^{1/2}$$

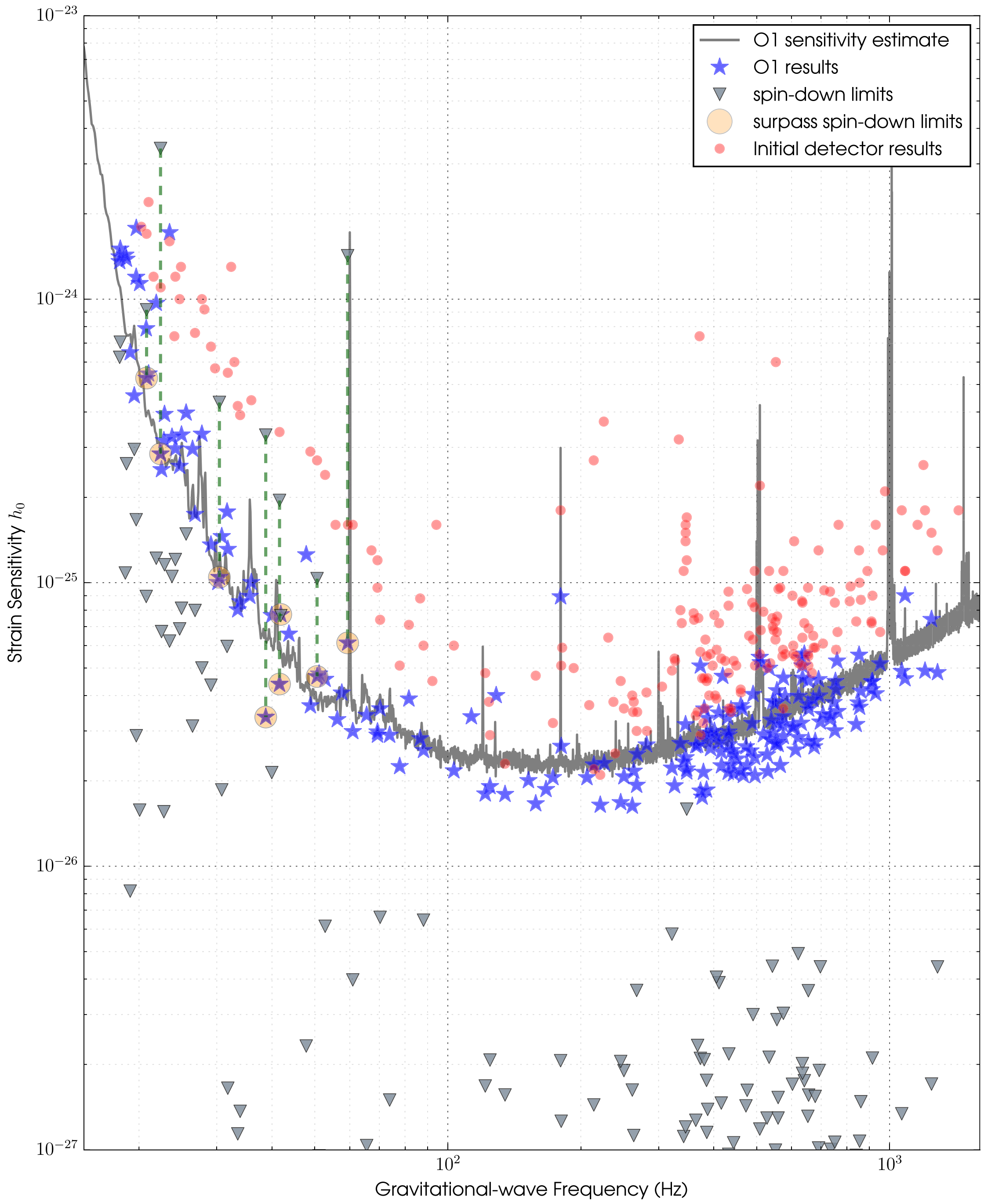
As in the case of the Crab, this is likely to be a very optimistic upper limit on the actual deformation.

The raw GW signal is feeble. We have

$$h \approx 8 \times 10^{-28} \left(\frac{\epsilon}{10^{-6}} \right) \left(\frac{f}{100 \text{ Hz}} \right)^2 \left(\frac{10 \text{ kpc}}{r} \right)$$

This is far too weak to be detectable, but the effective amplitude of a continuous wave signal improves as the square-root of the observation time.

Still need long observations, but many objects have known frequency and position so we can (at least) target searches.



Highlight results from O1:

- Crab pulsar: less than a fraction of a % of energy loss goes into GWs
- Spin-down limit beaten for 8 pulsars
- strongest constraint $\varepsilon \approx 10^{-8}$ represents an astonishing symmetry?)

Let us now consider the physics. What level of asymmetry can NS sustain?

Let us assume the asymmetry is due to elastic strain in the star's crust.

We compare a strained configuration, with oblateness ε_0, to a relaxed shape, with $\varepsilon<\varepsilon_0$.

Note: This is a "hand-of-God" argument.

The total energy of the star takes the form

$$E = E_0 + \frac{J^2}{2I} + A\epsilon^2 + B(\epsilon - \epsilon_0)^2$$

Accounting for kinetic energy, changes in the potential energy due to the shape and the strain. Minimising (at fixed J) we get

$$\epsilon = \frac{I_0\Omega^2}{4(A+B)} + \frac{B}{A+B}\epsilon_0 \equiv \mathrm{e}_\Omega + b\epsilon_0$$

Where

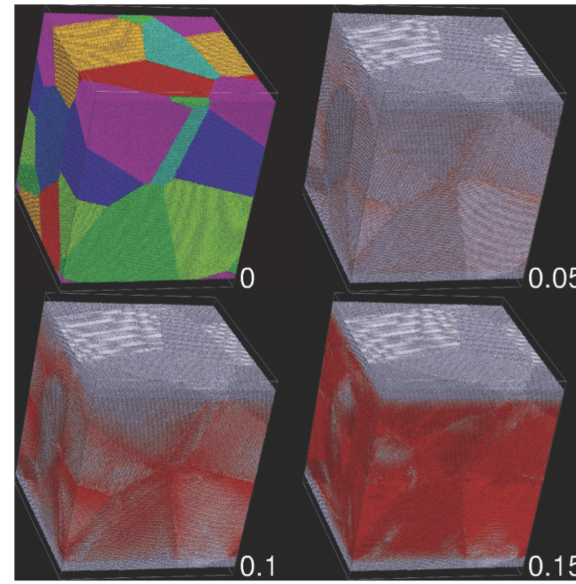
$$b \approx \frac{25}{2}\left(\frac{c^2 R}{GM}\right)\frac{\Delta R}{R}\frac{\mu}{\rho c^2} \approx 6\times 10^{-5}$$

for a "typical" model.

Since b<<1, a real NS will mainly be deformed by the centrifugal force.

However, ε and ε_0 can differ only by a factor that encodes the breaking strain of the crust material. In essence, we have the maximum deformation:

$$\epsilon \approx \frac{\mu V_{\rm crust}}{GM^2/R} \times u_{\rm break} \approx 10^{-6}\left(\frac{u_{\rm break}}{10^{-1}}\right)$$



[Horowitz and Kadau 2009]

The breaking strain is difficult to estimate, even for terrestrial materials.

Fairly recent molecular dynamics simulations suggest that the breaking strain $u_{\rm break}$ is larger than expected, around 0.1.

In essence, the crust is super-strong!

Key questions:

Why is the star deformed in the first place?

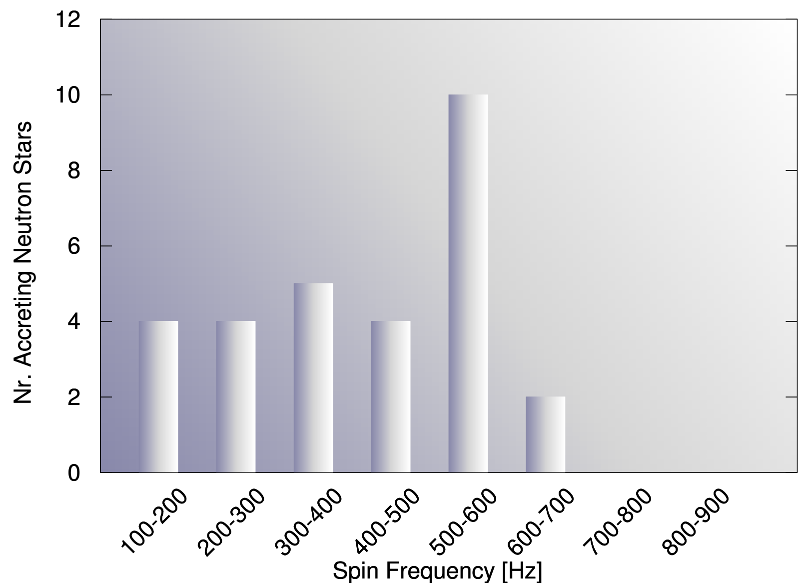
Do real neutron star mountains reach breaking strain or are stresses released gradually (through plastic flow)?

How does the crust “yield”?

Comment: In order to model the elastic strain we need to keep track of the relaxed (unstrained) configuration - brings in evolutionary aspects.

As soon as we start thinking about evolution, it is natural to consider accreting NS. In these systems asymmetries of the accretion flow may help generate the deformations we need.

Moreover, there is a mystery: Why do the observed systems not spin as fast as they could do according to theory?



The fastest (known) NS in a low-mass X-ray binary, 4U 1608, spins at 620 Hz. In order to explain this we may need;

– a non-standard accretion torque

– additional GW spin-down (mountains, r-modes, B-field)

Let us first estimate the maximum rotation rate. We can do this using the so-called Roche approximation (essentially pretending that the gravitational potential remains that of a sphere). Then, a uniformly rotating star is determined by

$$\frac{1}{\rho}\nabla_i p = -\nabla_i \left(\Phi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta \right) \qquad \Phi \approx -\frac{GM}{r}$$

Introducing the enthalpy, this leads to

$$h + \Phi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta = \text{ constant } = H$$

Evaluate at the pole (with *h*=0 at the surface) to determine *H*;

$$H = -\frac{GM}{R_p}$$

Now the maximum rotation rate is reached when the surface rotates at the Kepler frequency of a orbiting particle. At the equator when then have

$$\Omega_K^2 = \frac{GM}{R_e^3} \quad \Longrightarrow \quad -\frac{3}{2}\frac{GM}{R_e} = -\frac{GM}{R_p} \longrightarrow \frac{R_e}{R_p} = \frac{3}{2}$$

And we arrive at the (surprisingly accurate) approximation

$$\Omega_K \approx \left(\frac{2}{3}\right)^{3/2}\sqrt{\frac{GM}{R_p^3}} \approx \left(\frac{2}{3}\right)^{3/2}\sqrt{\frac{GM}{R_0^3}} \approx \frac{2}{3}\sqrt{\pi G \rho_0}$$

Typically over 1 kHz

Next, let us ask how large the NS deformation has to be in order for GW emission to balance the accretion.

Assuming for simplicity that the can ignore the star's magnetic field, accretion leads to a torque;

$$\dot{J} = \dot{M}\sqrt{GMR}$$

Relating energy and energy losses through

$$\dot{E} = \Omega \dot{J}$$

And using the quadrupole formula result for the GW emission, we find that we need

$$\epsilon = 4 \times 10^{-8} \left(\frac{\dot{M}}{10^{-9} M_\odot/\mathrm{yr}} \right)^{1/2} \left(\frac{300 \text{ Hz}}{\nu_s} \right)^{5/2}$$

We already know that a NS can sustain this level of deformation. So accreting NS may be relevant GW sources.

However;

- interaction between the star's magnetosphere and the accretion flow may significantly affect the torque (and can lead to spin equilibrium without GWs)
- the GW signal will be weak and as the systems are variable it is likely to be difficult over a long enough stretch of data
- but... there may be some indicative evidence from X-ray timing.

We have discussed the largest permissible mountain, but we would actually like to know the **smallest** deformation we should expect.

As usual in astrophysics, the answer may be “the magnetic field”.

Simple estimate (based on energetics) leads to

$$\epsilon \sim \frac{\int B^2\, dV}{GM^2/R} \sim 10^{-12} \left(\frac{B}{10^{12}\,\mathrm{G}} \right)^2$$

If protons form type II superconductor (as expected), the magnetic field is confined to fluxtubes. This increases the tension by a factor of H_c/B, where $H_c \sim 10^{15}$ G, and we get

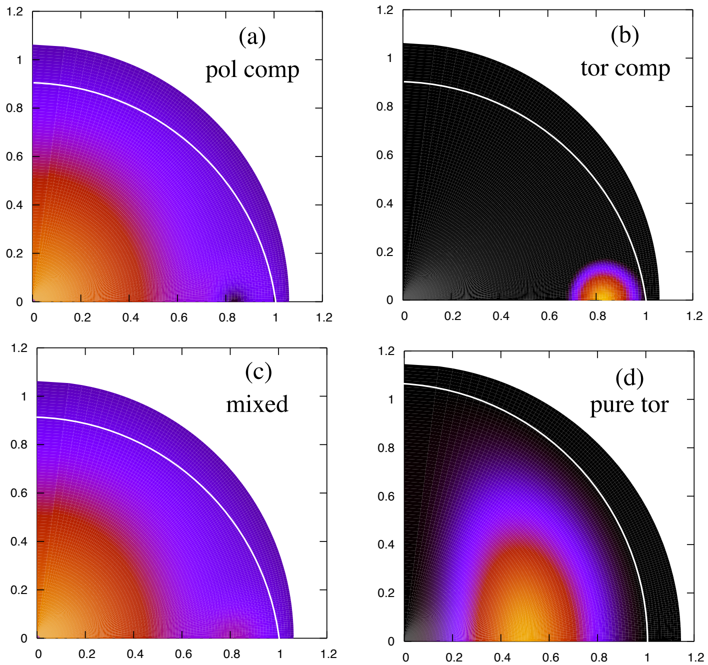
$$\epsilon \sim 10^{-9} \frac{B}{10^{12}\,\mathrm{G}}$$

The GW emission from known pulsars would still not be detectable…

The “smallest” NS mountain may simply be too small.

More detailed calculations (pretty much) give the same results.

In general, the modelling of magnetic deformations is tricky because we need the internal field configuration (which is unknown).



Nevertheless, it is worth taking a brief look at this problem.

There is a competition between the poloidal (which makes star oblate) and toroidal (which makes it prolate) components.

The ratio between the two is not known, but numerical models tend to find that the toroidal contribution is weaker.

Although… for magnetars it is usually "assumed" that the opposite is true. So something is "wrong".

Difficult to reconcile within the standard assumptions. Hydromagnetic equilibrium follows from:

$$\frac{\nabla p}{\rho} + \nabla \Phi = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi\rho} \qquad \nabla \times \left[\frac{\boldsymbol{B} \times (\nabla \times \boldsymbol{B})}{\rho} \right] = 0$$

- for barotropes we arrive at the Grad-Shafranov equation,
- for non-barotropes we may use "whatever field we like", but the system will not be in chemical equilibrium.

Note: Few (if any) known equilibria are actually stable!