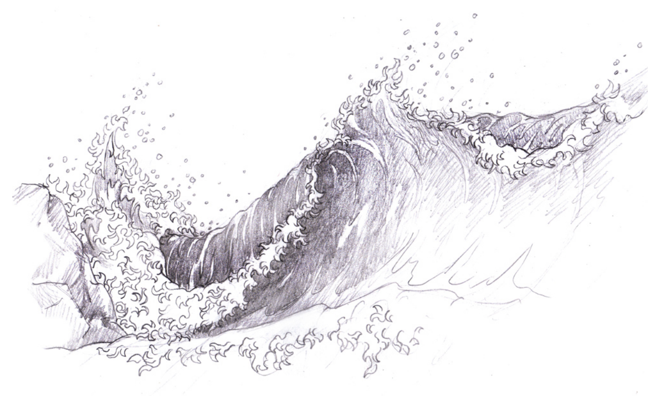
Lecture 1

What's the matter?

St Andrews summer school 2017



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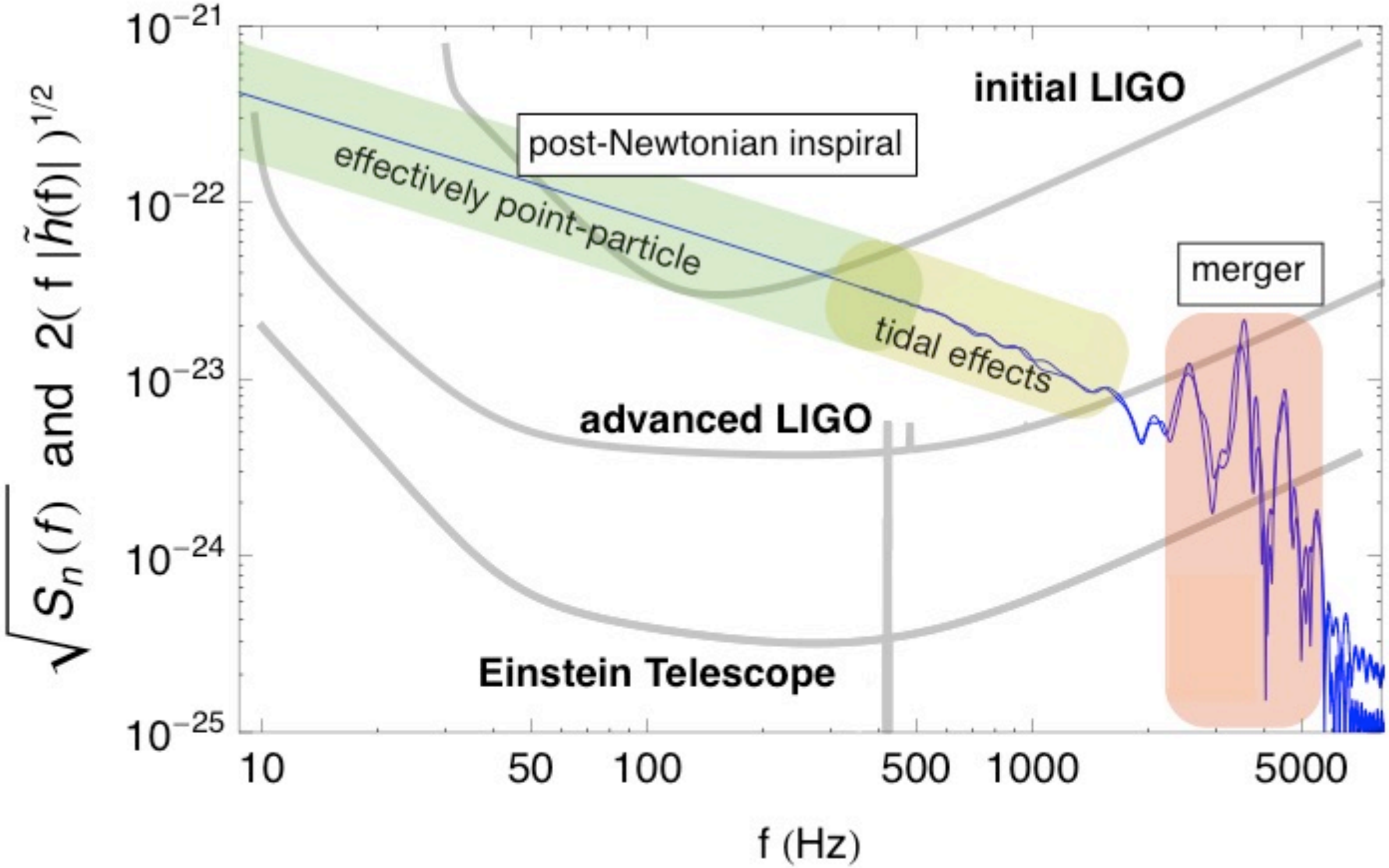
$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

post-Newtonian ➔ perturbation theory ➔ numerical relativity



Consider the weak-field limit, and write the metric as a small deviation from the Minkowski spacetime;

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O\left(h^2\right)$$

We will assume that *h* is small (in a suitable sense) and keep only linear terms in all calculations. It follows that

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + O\left(h^2\right)$$

Carrying out the Ricci contraction on the linearised Riemann tensor, and introducing

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$$

Noting that,

$$\bar{h} = \eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = h - 2h = -h$$

we see that this variable simply has the sign of the trace reversed. (At a deeper level, this variable is motivated by the form of the Einstein tensor.)

If we also adopt the **Lorenz gauge**

$$\partial^\alpha \bar{h}_{\alpha\beta} = 0 \quad \Rightarrow \quad \partial^\alpha h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\partial^\alpha h = 0$$

we find that

$$R_{\alpha\beta} = -\frac{1}{2}\left(\Box \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\Box h\right) \quad \Rightarrow \quad R = -\frac{1}{2}\left(\Box \bar{h} + 2\Box h\right) = -\frac{1}{2}\Box h$$

and the Einstein equations become

$$\boxed{\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta}}$$

where the right-hand side encodes only asymmetric contributions to the stress-energy tensor. We can use the standard retarded Green's function to integrate the wave equation. Thus, we arrive at the quadrupole formula (TT=Transverse+Traceless)

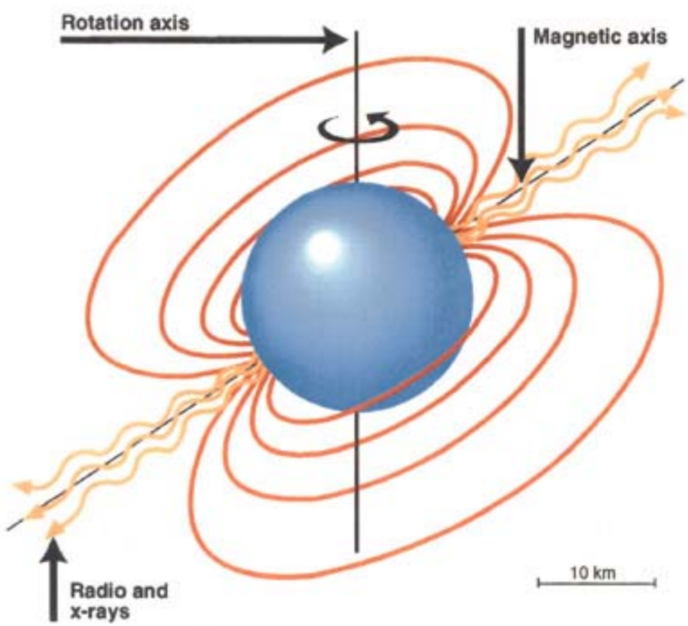
$$\bar{h}_{jk}^{\mathrm{TT}} = \frac{2G}{rc^4}\ddot{\mathcal{I}}_{jk}^{\mathrm{TT}}(t-r) \qquad \mathcal{I}_{jk} \equiv \int \rho\left(x_j x_k - \frac{1}{3}r^2\delta_{jk}\right) d^3x$$

Averaging over several wavelengths we also have the luminosity

$$\frac{dE}{dt} = \frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}^{jk}\right\rangle$$

Main lesson: Need to keep track of the acceleration of matter!

In principle, neutron stars are cosmic laboratories of extreme physics. They are expected to be important GW sources, and we hope to (eventually) probe matter at supranuclear densities.



mergers:

the inspiral chirp provides a clean GW signal carrying information about the system, while the merger phase probes strong field gravity

supernova core collapse:

the birth of a NS may lead to a GW burst

“mountains”:

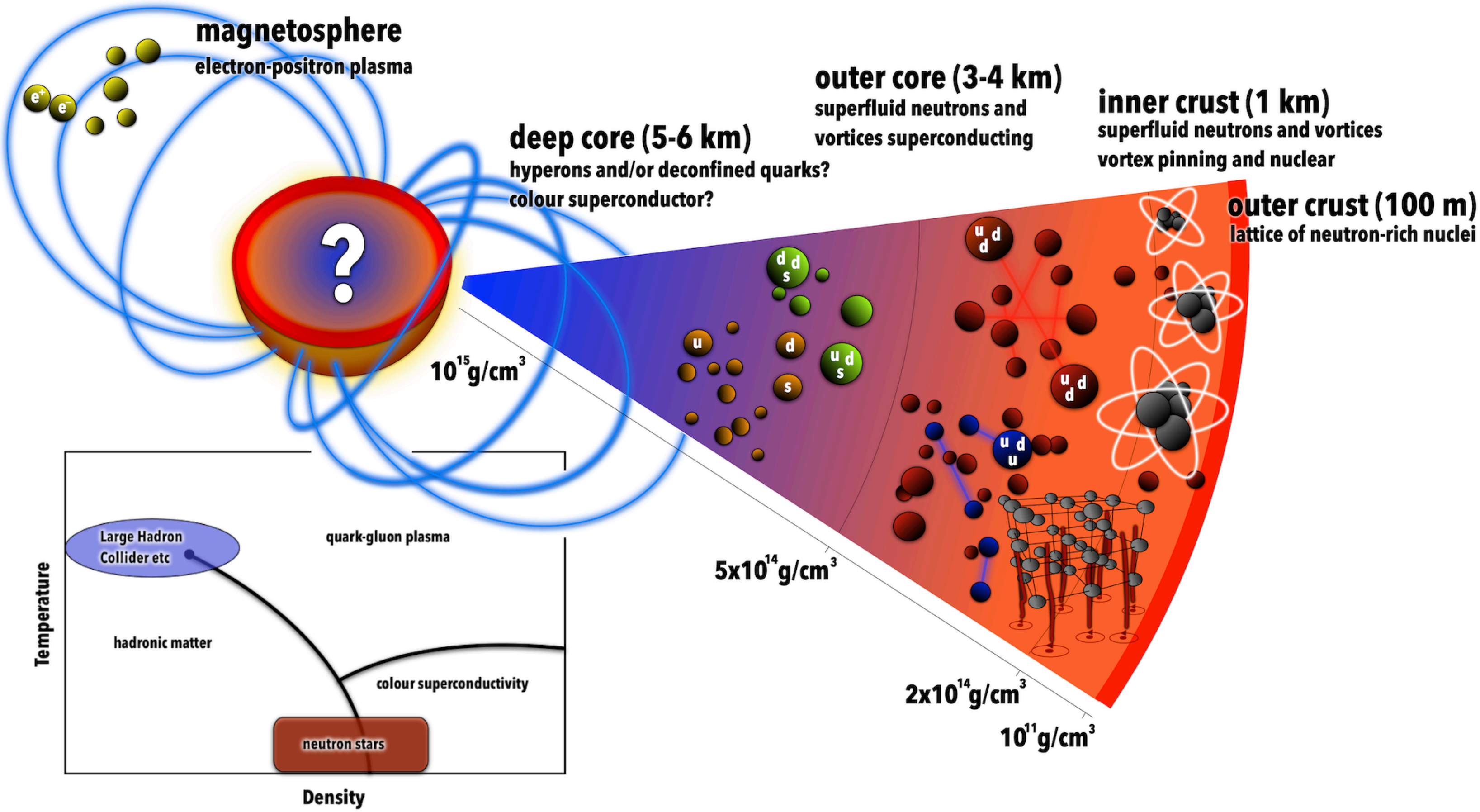
crustal asymmetries lead to GWs at twice the spin frequency

oscillations/instabilities:

fast spinning NS may suffer both dynamical bar-mode and secular instabilities (r-modes?)

Each mechanism is easy to understand “in principle” but difficult to model “in practice”.

Let us explore why this is so...



All four fundamental forces at play:

Gravity, holds the star together (gravitational waves?)

Electromagnetism, makes pulsars pulse and magnetars flare (radio/X-rays)

Strong interaction, determines the internal composition

Weak interaction, affects reaction rates - cooling and internal viscosity

The **equation of state** is the main diagnostic of dense matter interactions.

Each model generates a unique mass-radius relation, predicting a characteristic radius for a range of masses and a maximum mass above which a neutron star collapses to a black hole.

Constrain the physics by combining data from different observational channels.

So far:

Orbital data for binaries provide accurate masses.

Surface phenomena constrain the radius of a 1.4 $M_{\odot}$ star to 10-13 km.

Keep in mind:

- first principles calculations are challenging,
- astrophysics may do better than upcoming nuclear physics experiments (e.g. PREX).

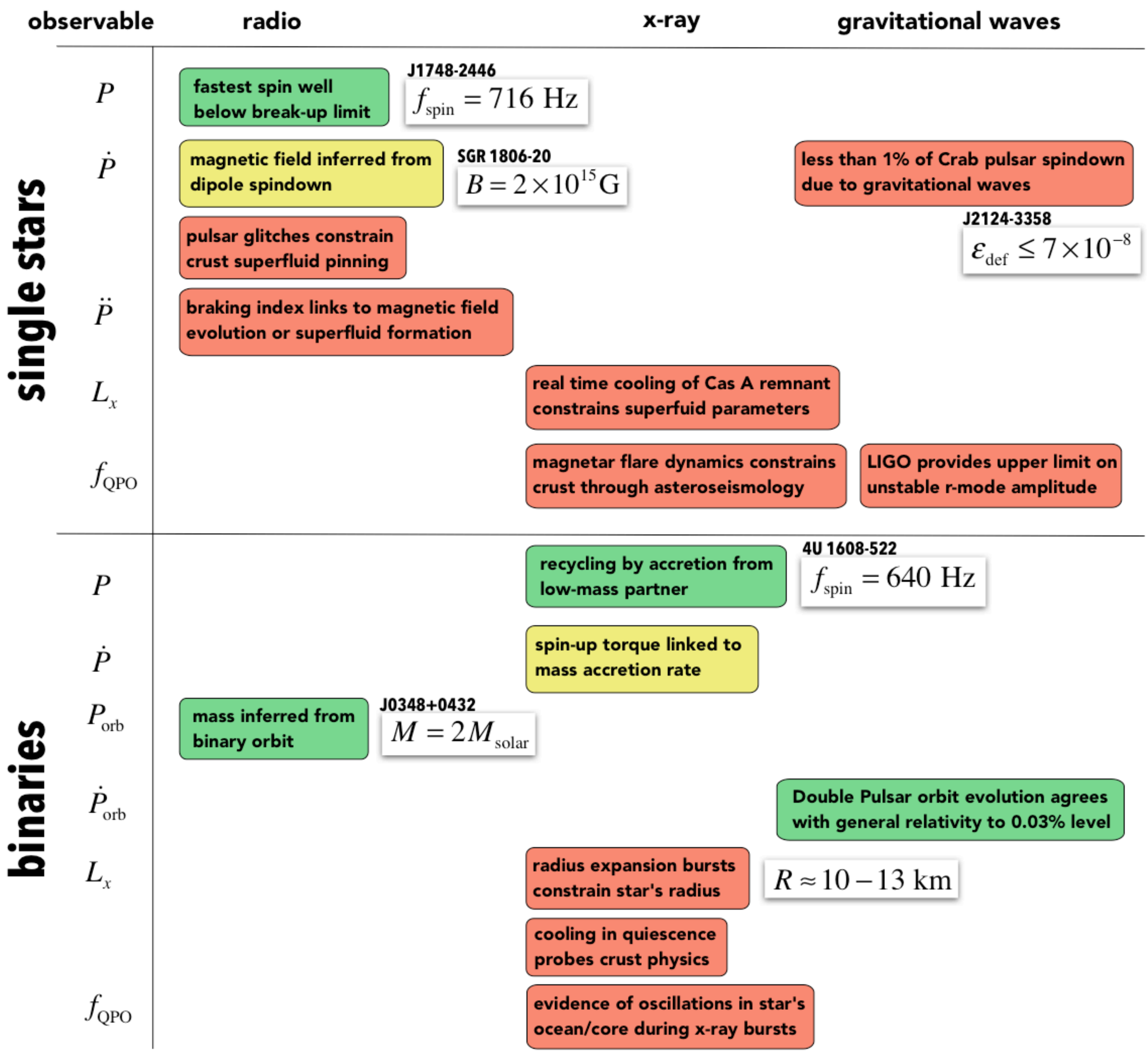
Soon-ish:

NASAs NICER mission will provide an "accurate" data point.

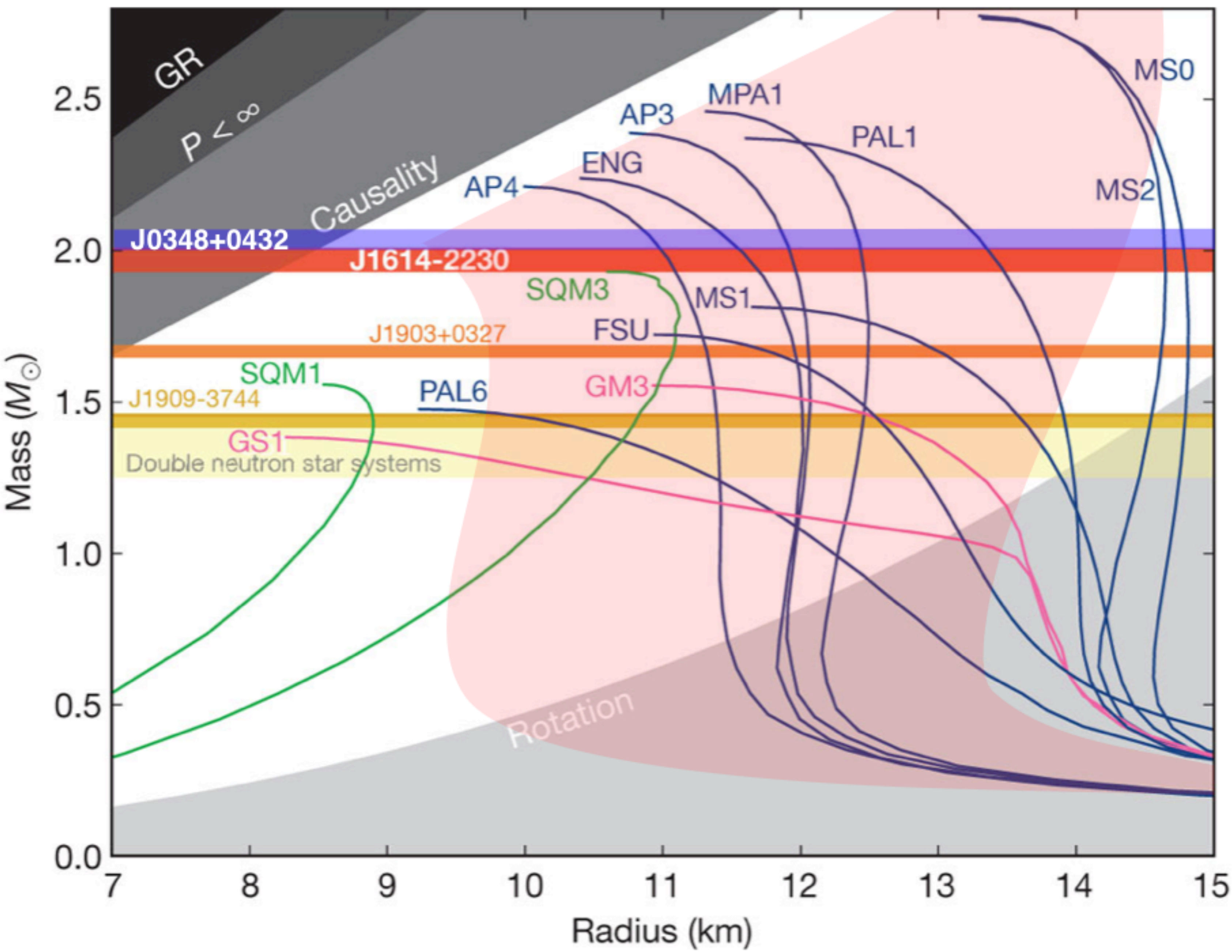
SKA will provide a much larger sample of neutron star masses.

Athena will add to the wealth of surface data (Chandra, XMM, NuSTAR).

Need a precision X-ray timing mission (like LOFT) to study burst dynamics and magnetar seismology.



observable
radio
x-ray
gravitational waves
single stars
P
fastest spin well below break-up limit
J1748-2446
$f_{\mathrm{spin}} = 716\ \mathrm{Hz}$
$\dot{P}$
magnetic field inferred from dipole spindown
SGR 1806-20
$B = 2\times10^{15}\,\mathrm{G}$
less than 1% of Crab pulsar spindown due to gravitational waves
J2124-3358
$\varepsilon_{\mathrm{def}} \le 7\times10^{-8}$
pulsar glitches constrain crust superfluid pinning
$\ddot{P}$
braking index links to magnetic field evolution or superfluid formation
L_x
real time cooling of Cas A remnant constrains superfluid parameters
f_{QPO}
magnetar flare dynamics constrains crust through asteroseismology
LIGO provides upper limit on unstable r-mode amplitude
binaries
P
recycling by accretion from low-mass partner
4U 1608-522
$f_{\mathrm{spin}} = 640\ \mathrm{Hz}$
$\dot{P}$
spin-up torque linked to mass accretion rate
P_{orb}
mass inferred from binary orbit
J0348+0432
$M = 2M_{\mathrm{solar}}$
$\dot{P}_{\mathrm{orb}}$
Double Pulsar orbit evolution agrees with general relativity to 0.03% level
L_x
radius expansion bursts constrain star's radius
$R \approx 10-13\ \mathrm{km}$
cooling in quiescence probes crust physics
f_{QPO}
evidence of oscillations in star's ocean/core during x-ray bursts



GR
$P < \infty$
Causality
J0348+0432
J1614-2230
J1903+0327
J1909-3744
Double neutron star systems
Rotation
SQM1
SQM3
GS1
GM3
PAL6
FSU
MS1
AP4
ENG
AP3
MPA1
PAL1
MS2
MS0
Mass ($M_\odot$)
Radius (km)
0.0
0.5
1.0
1.5
2.0
2.5
7
8
9
10
11
12
13
14
15

Focussing on the fluid dynamics, we need the equations of motion. These follow from

$$\nabla_\alpha T^{\alpha\beta} = 0$$

Most GW models are based on the notion of a perfect fluid;

$$T^{\alpha\beta} = (p+\varepsilon)u^\alpha u^\beta + pg^{\alpha\beta}$$

Note: this does not allow for heat flux, charge currents, elasticity, superfluidity...

Also: We are using a fibration of spacetime with u^α the four-velocity associated with a "fluid element".

We get

$$\nabla_\alpha T^{\alpha\beta} = \nabla_\alpha\left[\left(p+\varepsilon\right)u^\alpha u^\beta + pg^{\alpha\beta}\right] =$$

$$= u^\beta \nabla_\alpha\left[\left(p+\varepsilon\right)u^\alpha\right] + \left(p+\varepsilon\right)u^\alpha \nabla_\alpha u^\beta + g^{\alpha\beta}\nabla_\alpha p = 0$$

Contract this with u_β we have

$$\underbrace{\left(u_\beta u^\beta\right)}_{=-1}\nabla_\alpha\left[\left(p+\varepsilon\right)u^\alpha\right] + \underbrace{u_\beta g^{\alpha\beta}}_{=u^\alpha}\nabla_\alpha p = -u^\alpha \nabla_\alpha \varepsilon - \left(p+\varepsilon\right)\nabla_\alpha u^\alpha = 0$$

This result encodes the conservation of energy.

To see that the equation "makes sense", we need a little bit of thermodynamics… Assuming that $\varepsilon=\varepsilon(n)$, we have the **chemical potential**

$$\mu = \frac{d\varepsilon}{dn} \quad \text{and the identity} \quad \mu n = p + \varepsilon \quad \Rightarrow$$

$$\mu u^\alpha \nabla_\alpha n + \mu n \nabla_\alpha u^\alpha = \mu \nabla_\alpha \left(n u^\alpha \right) = 0 \quad \Rightarrow \quad \nabla_\alpha n^\alpha = 0$$

The particle flux n^α is conserved.

Next, consider the projection orthogonal to the four velocity;

$$\perp^\alpha_\gamma u^\gamma = \left(\delta^\alpha_\gamma + u^\alpha u_\gamma \right) u^\gamma = 0$$

gives

$$\perp_{\gamma\beta} \nabla_\alpha T^{\alpha\beta} = \underbrace{\perp_{\gamma\beta} u^\beta}_{=0} \nabla_\alpha \left[\left(p + \varepsilon \right) u^\alpha \right] + \perp_{\gamma\beta} \left(p + \varepsilon \right) u^\alpha \nabla_\alpha u^\beta + \underbrace{\perp_{\gamma\beta} g^{\alpha\beta}}_{=\perp^\alpha_\gamma} \nabla_\alpha p = 0$$

Introducing the four acceleration

$$a^\alpha = u^\beta \nabla_\beta u^\alpha$$

we can write the equation in a form that reminds us of Newton's second law;

$$\left(p + \varepsilon \right) a_\gamma = - \perp^\alpha_\gamma \nabla_\alpha p$$

That is, pressure gradients drive changes in the four-acceleration.

Perfect fluids do not "move" on geodesics…

This is nice, but... How **accurate** is the perfect fluid assumption?

Any state-of-the-art model for neutron star dynamics must account for the fact that these are **multi-component** multi-fluid systems (the composition varies and there are relative flows – heat, charge currents, superfluids).

This requires “**beyond equilibrium**” equation of state information.

As example, consider the pressure perturbation for (cold) npe-matter;

$$p = p(n_n, n_p, n_e) \quad \Rightarrow$$

$$\delta p = n_n \delta\mu_n + n_p \delta\mu_p + n_e \delta\mu_e = \qquad \text{[1. definition]}$$

$$= n_n \delta\mu_n + n_p \left(\delta\mu_p + \delta\mu_e\right) \qquad \text{[2. charge neutrality]}$$

$$= n\left(1 - x_p\right)\delta\mu_n + n x_p \left(\delta\mu_p + \delta\mu_e\right) \qquad \text{[3. introduce proton fraction]}$$

$$= n\delta\mu_n + n x_p \left(\delta\mu_p + \delta\mu_e - \delta\mu_n\right) \qquad \text{[4. beta equilibrium]}$$

$$= n\delta\mu_n$$

Depending on the state of matter (normal/superfluid) and the regime (fast/slow reactions), one may have to keep track of many thermodynamical derivatives.

These may not be easy to infer from a tabulated equilibrium equation of state.

To see why this is important, consider the perturbed Euler equations for a Newtonian star (let c^2 become "infinite" in the previous equations);

$$\partial_t^2 \xi_i + \frac{1}{\rho} \nabla_i \delta p - \left(\frac{1}{\rho^2} \nabla_i p \right) \delta\rho + \nabla_i \delta\Phi = 0$$

We need the perturbed equation of state, which encodes how the matter reacts to being pushed out of equilibrium.

There are two "simple" limits. It is "natural" to use different variables depending on the circumstances. If nuclear reactions are faster than the dynamics, we can assume that the matter stays in equilibrium. Then we have $\beta = \mu_n - \mu_p - \mu_e$ and

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_\beta \delta\rho + \left(\frac{\partial p}{\partial \beta} \right)_\rho \delta\beta = \left(\frac{\partial p}{\partial \rho} \right)_\beta \delta\rho - \left(\frac{\partial p}{\partial \beta} \right)_\rho \xi^i \nabla_i \beta = \left(\frac{\partial p}{\partial \rho} \right)_\beta \delta\rho$$

If, on the other hand, the reactions are slow then the matter composition if "frozen" and

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_{x_p} \delta\rho + \left(\frac{\partial p}{\partial x_p} \right)_\rho \delta x_p = \left(\frac{\partial p}{\partial \rho} \right)_{x_p} \delta\rho - \left(\frac{\partial p}{\partial x_p} \right)_\rho \xi^i \nabla_i x_p$$

The difference may be subtle, but there are situations (e.g. tides) where it may turn out to be very important.