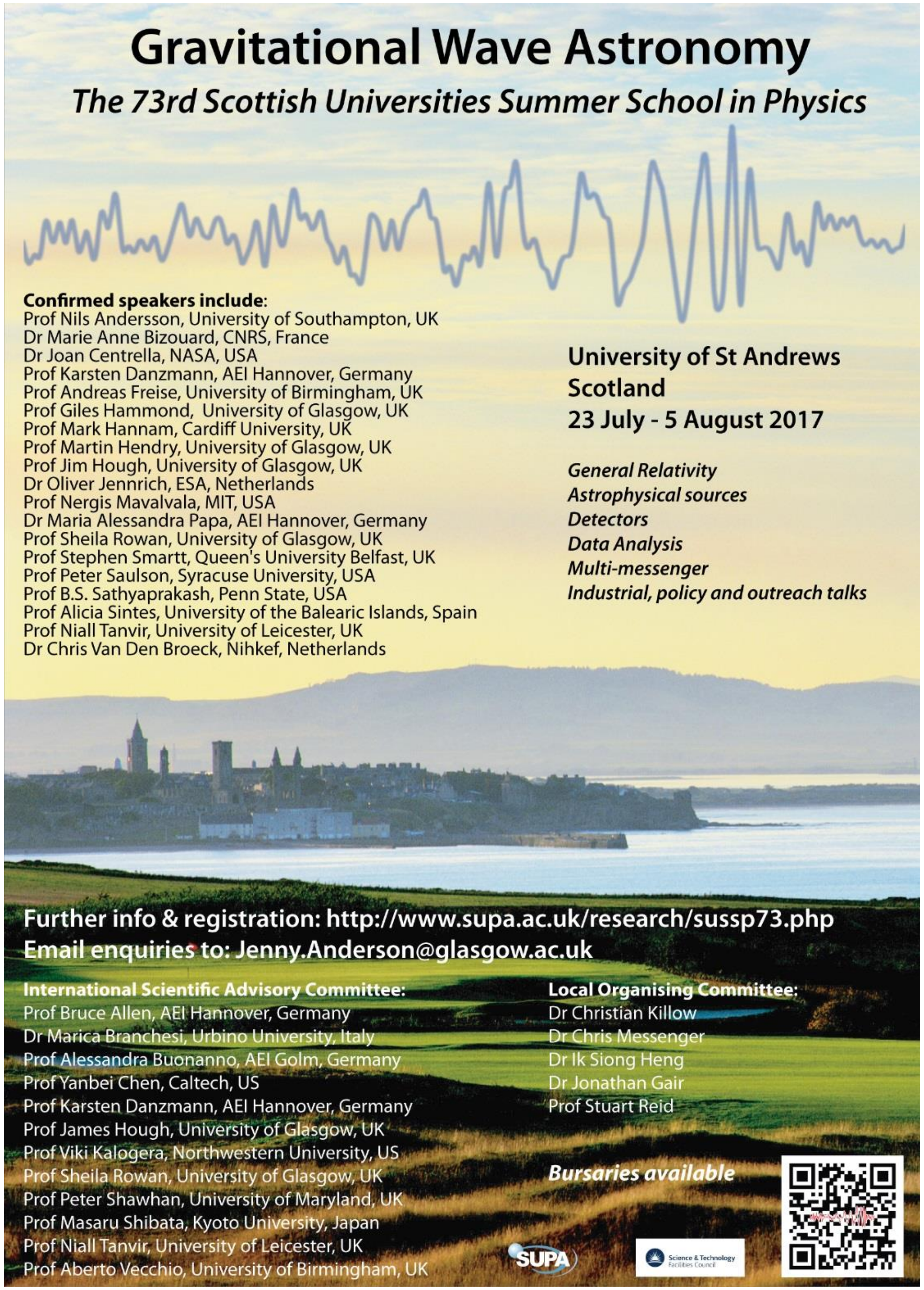
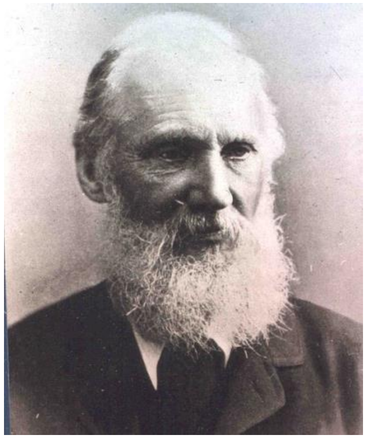
SUSSP73

An Introduction to General Relativity, Gravitational Waves and Detection Principles

Martin Hendry
University of Glasgow
Institute for Gravitational Research

July 2017





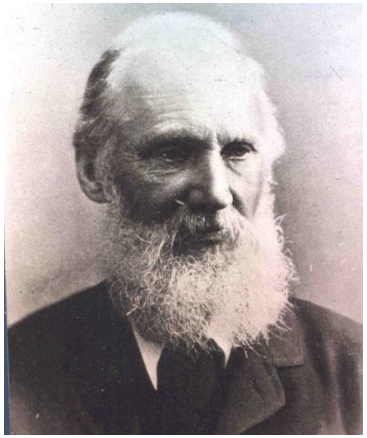
William Thompson
(Lord Kelvin)
1824 - 1907

University of Glasgow

IGR SUPA

“There is nothing new to be discovered in physics now.

All that remains is more and more precise measurement” (1900)



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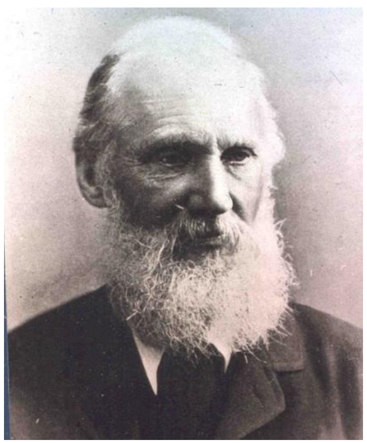

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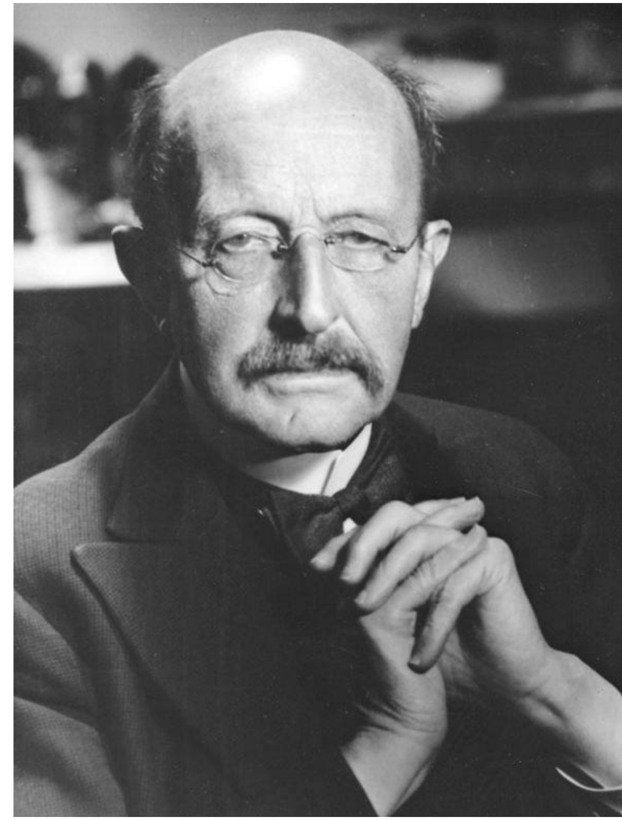

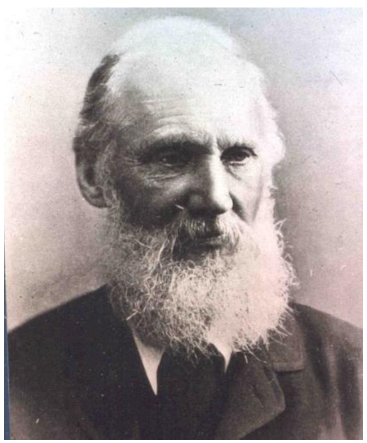

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VIA VERITAS VITA

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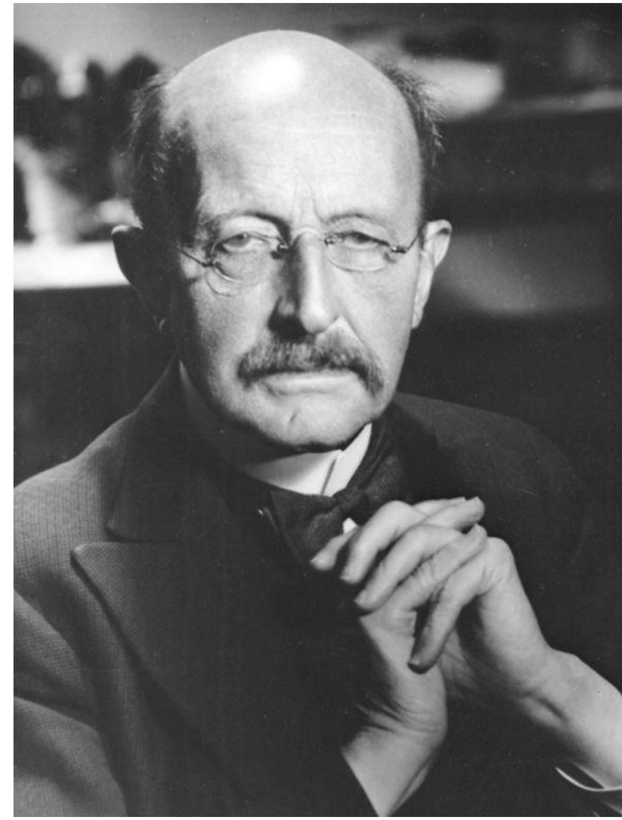

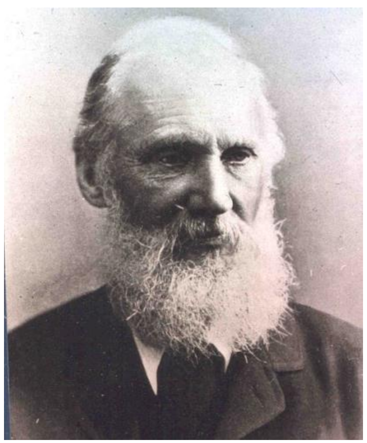

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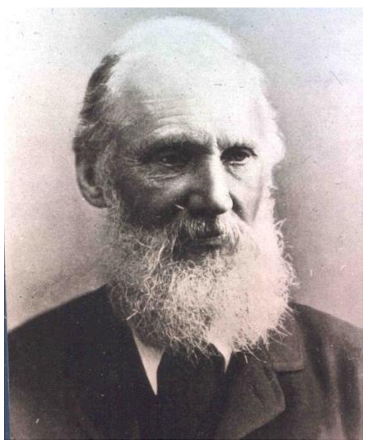
"In this field [physics] almost everything is already discovered, and all that remains is to fill a few holes."



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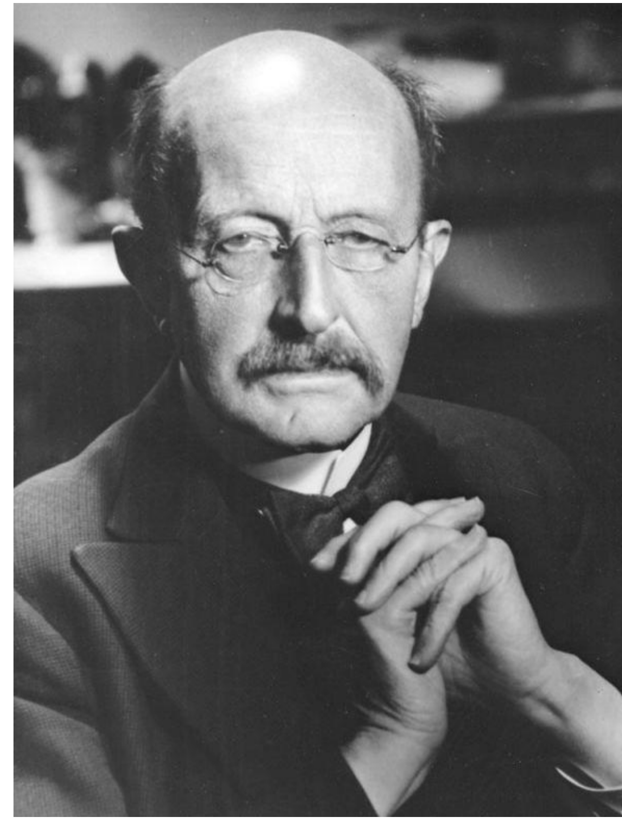

"In this field [physics] almost everything is already discovered, and all that remains is to *find* a few holes."

University of Glasgow

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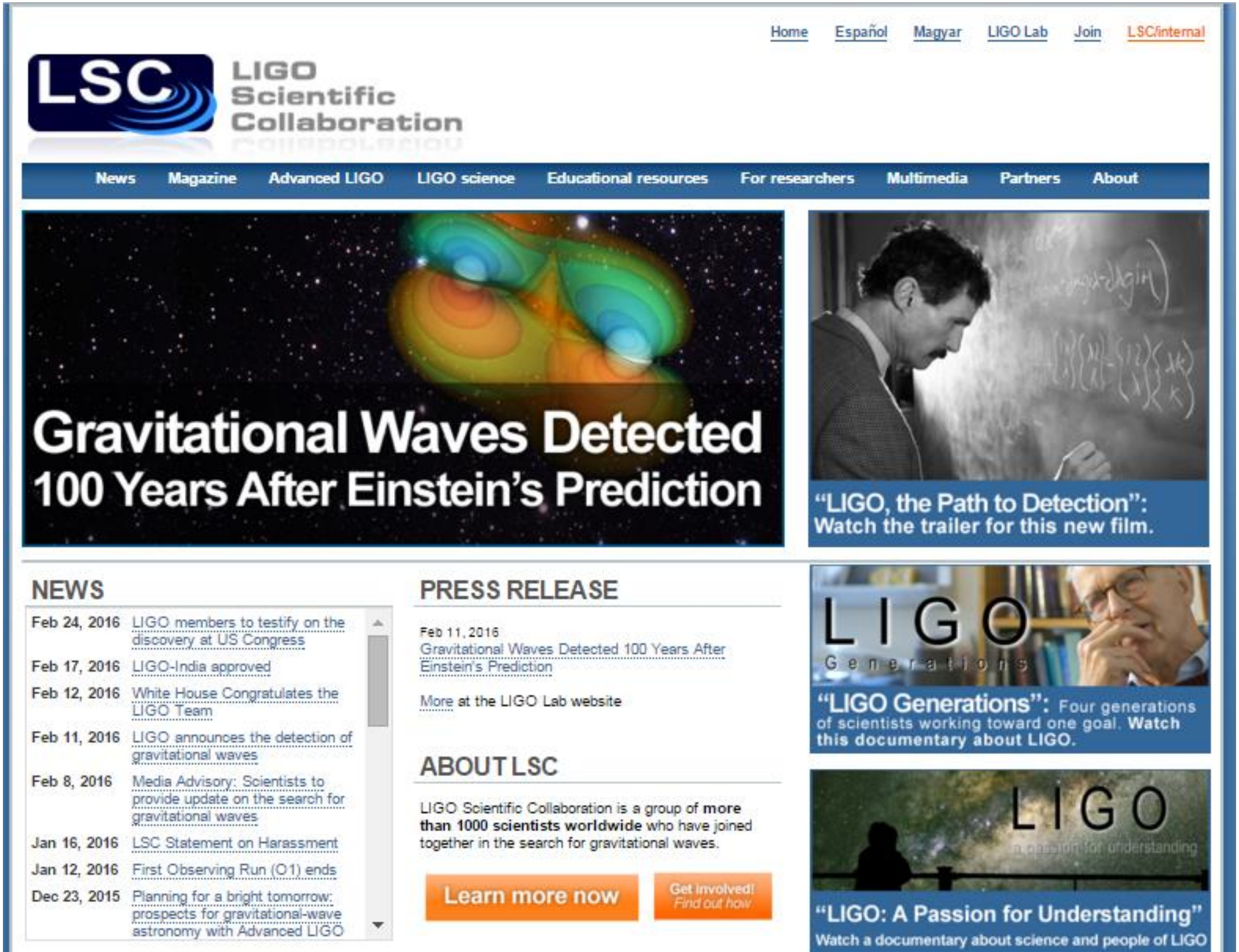
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"LIGO, the Path to Detection":
Watch the trailer for this new film.
NEWS
Feb 24, 2016 LIGO members to testify on the discovery at US Congress
Feb 17, 2016 LIGO-India approved
Feb 12, 2016 White House Congratulates the LIGO Team
Feb 11, 2016 LIGO announces the detection of gravitational waves
Feb 8, 2016 Media Advisory: Scientists to provide update on the search for gravitational waves
Jan 16, 2016 LSC Statement on Harassment
Jan 12, 2016 First Observing Run (O1) ends
Dec 23, 2015 Planning for a bright tomorrow: prospects for gravitational-wave astronomy with Advanced LIGO
PRESS RELEASE
Feb 11, 2016
Gravitational Waves Detected 100 Years After Einstein's Prediction
More at the LIGO Lab website
ABOUT LSC
LIGO Scientific Collaboration is a group of more than 1000 scientists worldwide who have joined together in the search for gravitational waves.
Learn more now
Get involved! Find out how
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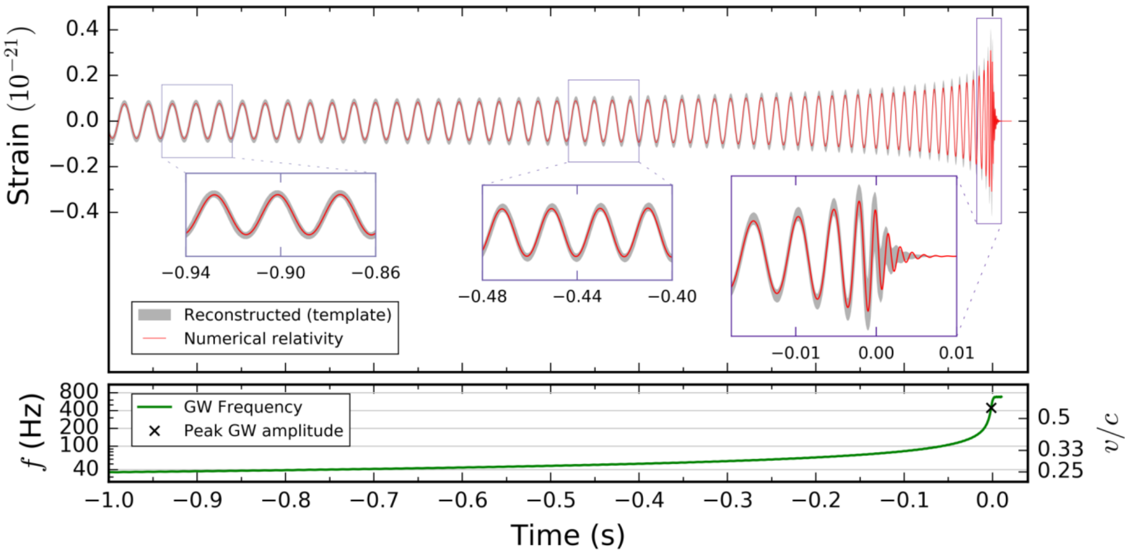
"LIGO Generations": Four generations of scientists working toward one goal. Watch this documentary about LIGO.
LIGO
a passion for understanding
"LIGO: A Passion for Understanding"
Watch a documentary about science and people of LIGO

GW151226: a late Christmas present

Binary black hole merger: ~14 and ~8 solar masses



https://www.ligo.caltech.edu/video/ligo20160615v3

- Lower SNR, detected via matched filtering
- Spent ~1 second in band; inspiral observed for much longer than with GW150914
- Peak GW strain of about 3.4 x 10^{-22}
- At least one of BHs has spin > 0.2

Abbott, et al.,
LIGO Scientific Collaboration and Virgo Collaboration,
'GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence'
Phys. Rev. Lett. 116, 241103 (2016)

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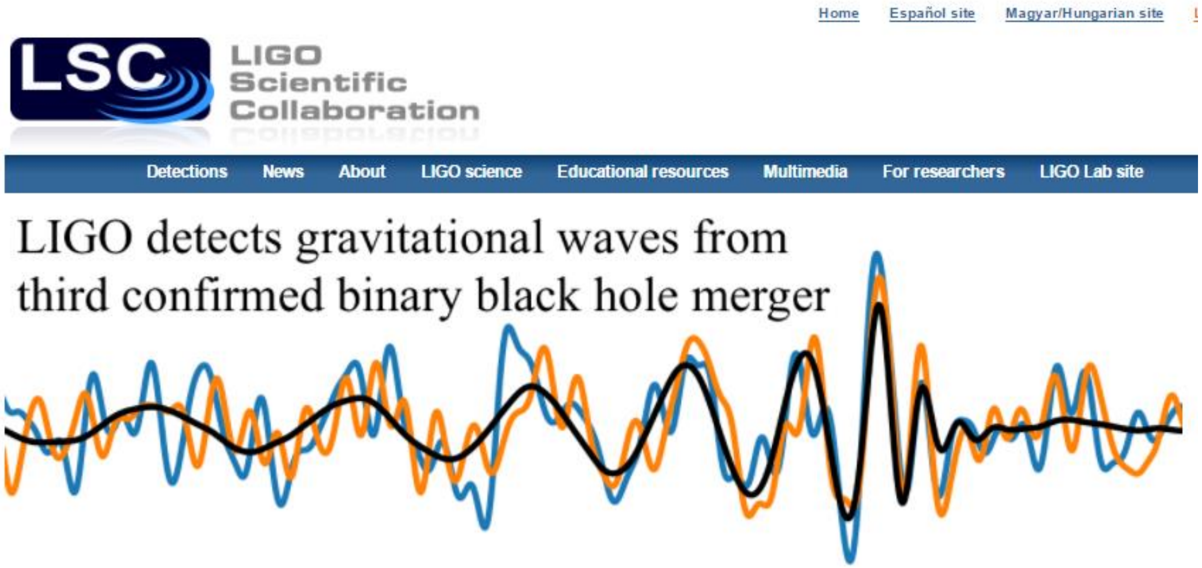
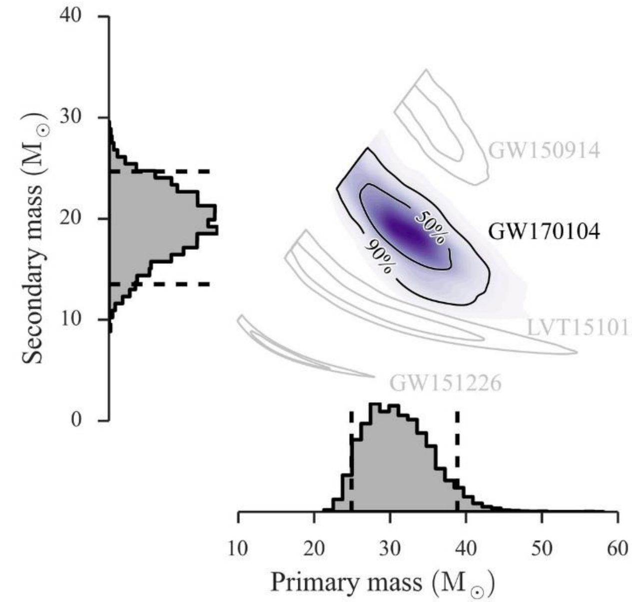
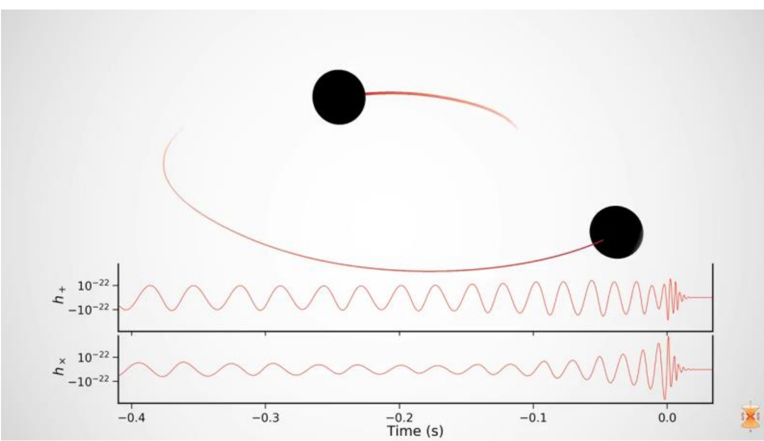
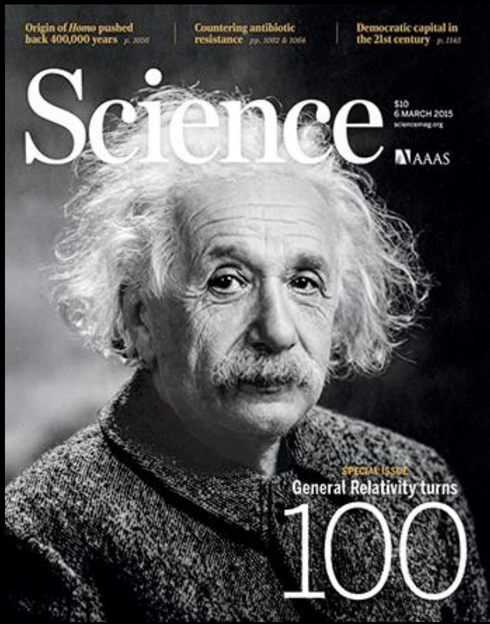




GW 170104 – Detected January 4th 2017

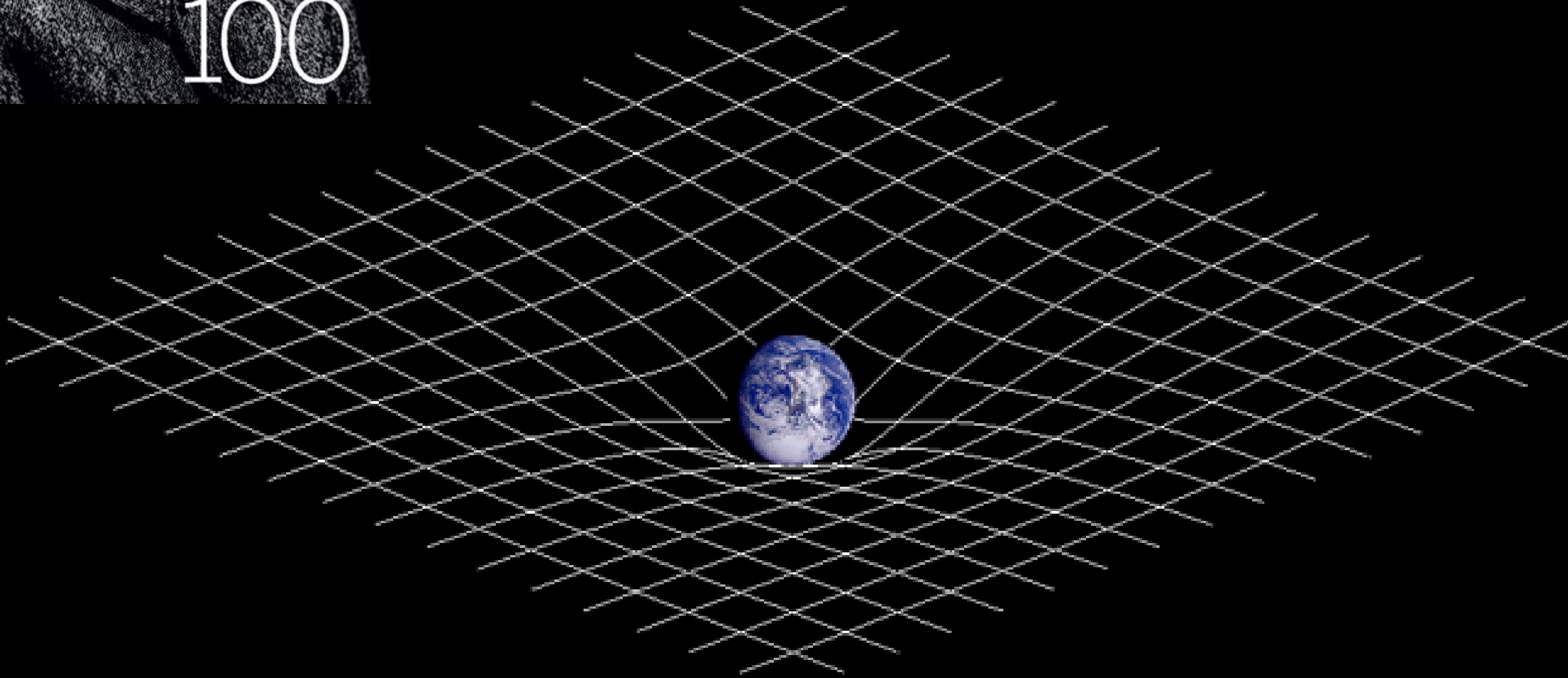
Announced June 1st 2017





Gravity in Einstein's Universe

"Spacetime tells matter how to move,
and matter tells spacetime how to curve"



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"The greatest feat of human thinking about nature, the most amazing combination of philosophical penetration, physical intuition and mathematical skill."   *Max Born*

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Spacetime curvature

Matter (and energy)

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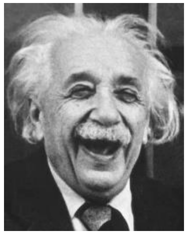
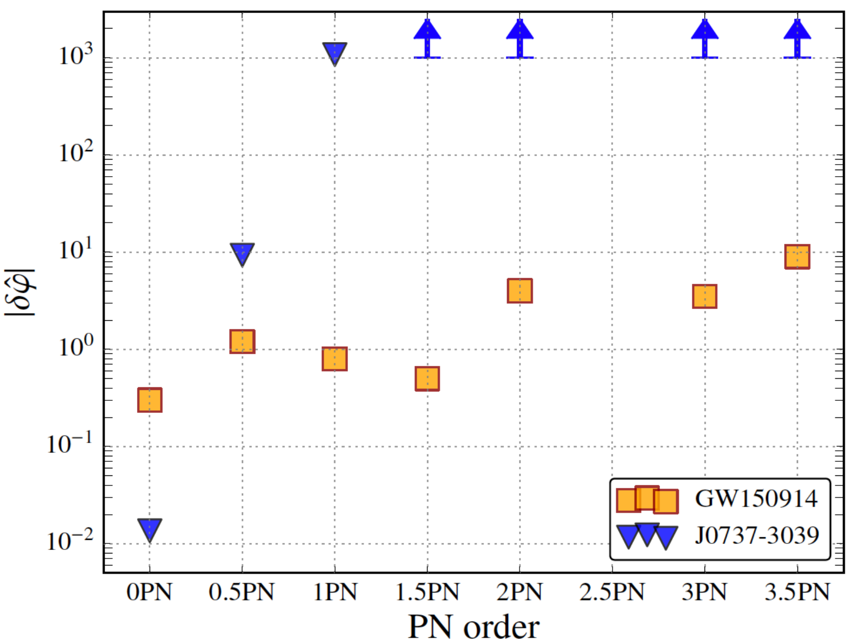


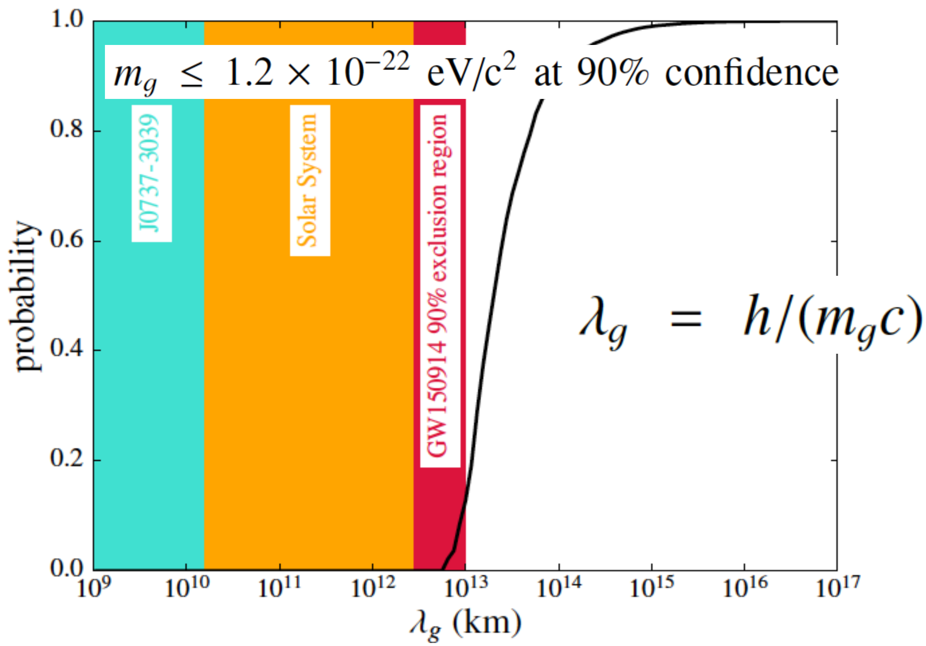
Does General Relativity really fit?

- GW150914 was the first observation of a binary black hole merger
- Our best test of GR in *the strong field, dynamical, nonlinear regime*
- Event better than the binary pulsar system PSR J0737-3039

Post-Newtonian_Approximation to GR



Compton Wavelength of the Graviton



Abbott, et al., LIGO Scientific Collaboration and Virgo Collaboration, “Tests of general relativity with GW150914”, http://arxiv.org/abs/1602.03841

- We now have **combined constraints** from all 3 confirmed detections

What we are going to (try to) cover

Introduction to GR

1. Foundations of general relativity
2. Introduction to geodesic deviation
3. A mathematical toolbox for GR
4. Spacetime curvature in GR
5. Einstein's equations

Gravitational Waves and detector principles

6. A wave equation for gravitational radiation
7. The Transverse Traceless gauge
8. The effect of gravitational waves on free particles
9. The production of gravitational waves

We are going to cram a lot of mathematics and physics into about 3 hours.

Two-pronged approach:

- Lecture slides presenting "highlights" and illustrations / examples
- Comprehensive lecture notes, providing a longer-term resource and reference source

SUPA Gravitational Wave Detection

University of Glasgow / SUSSP73

July 2017

An Introduction to General Relativity, Gravitational Waves and Detection Principles

Prof Martin Hendry

University of Glasgow, UK

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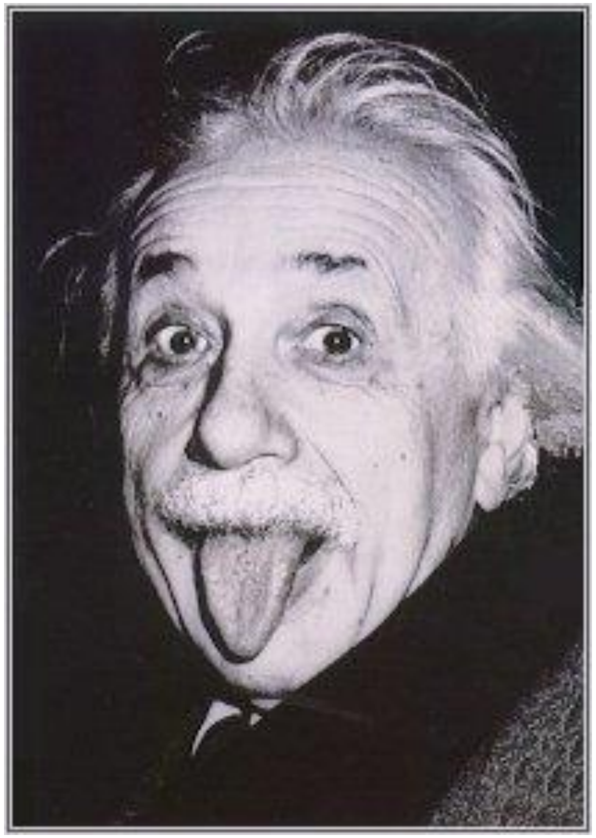
Copies of both will be available via the SUSSP73 website



"The hardest thing in the world to understand is the income tax"

1. Foundations of General Relativity (pgs. 6 – 12)

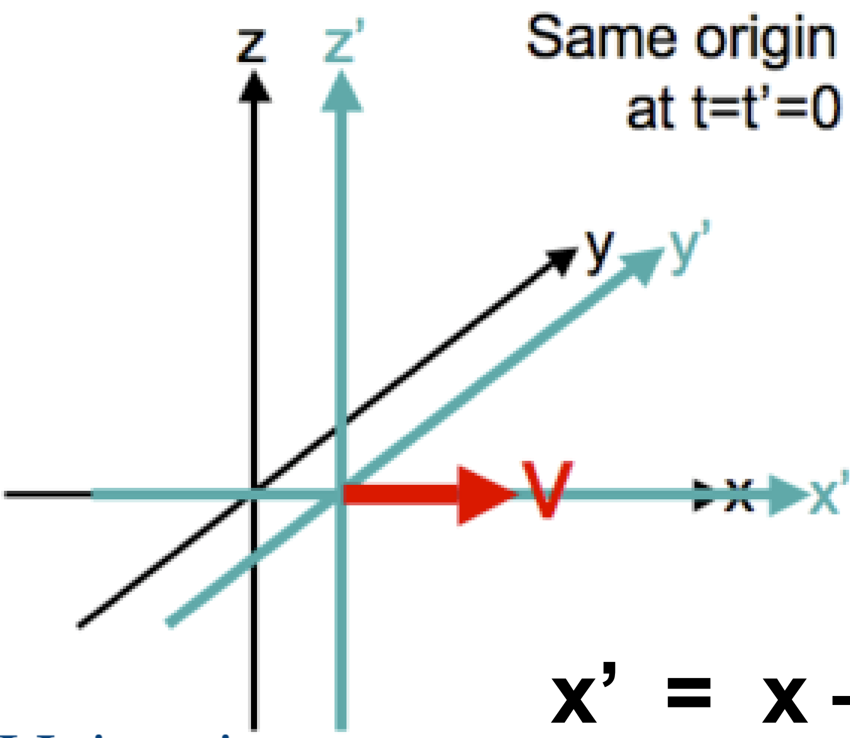
GR is a generalisation of Special Relativity (1905).

In SR Einstein formulated the laws of physics to be valid for all **inertial observers**

→ **Measurements of space and time relative to observer's motion.**

The world of Galileo and Newton (and everyday “common sense”...)

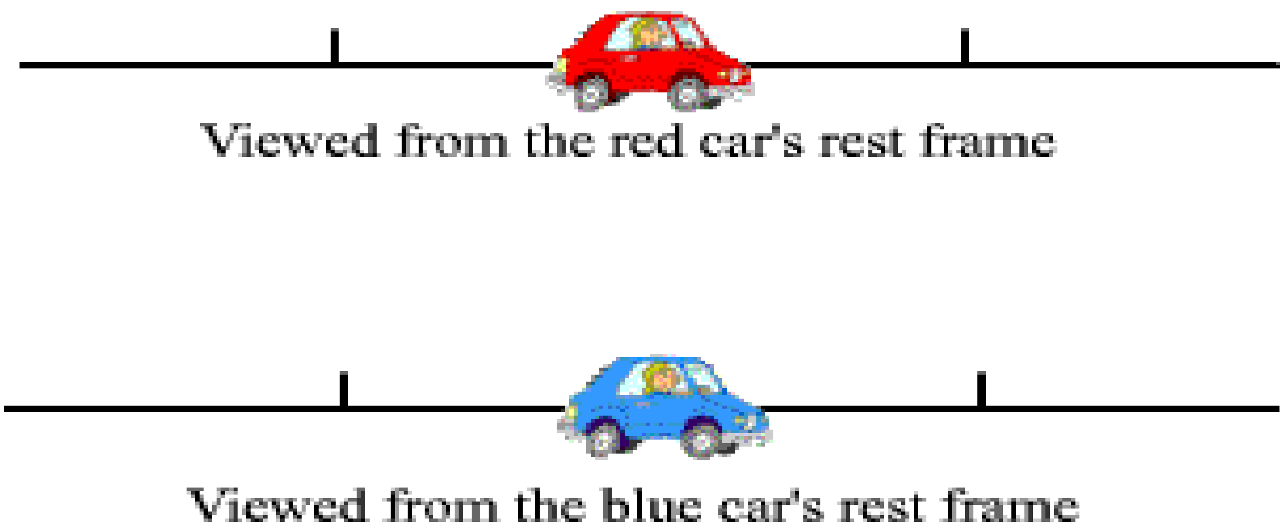
Working out how things look to different observers follows simple rules, in different *reference frames*



x' = x – v t

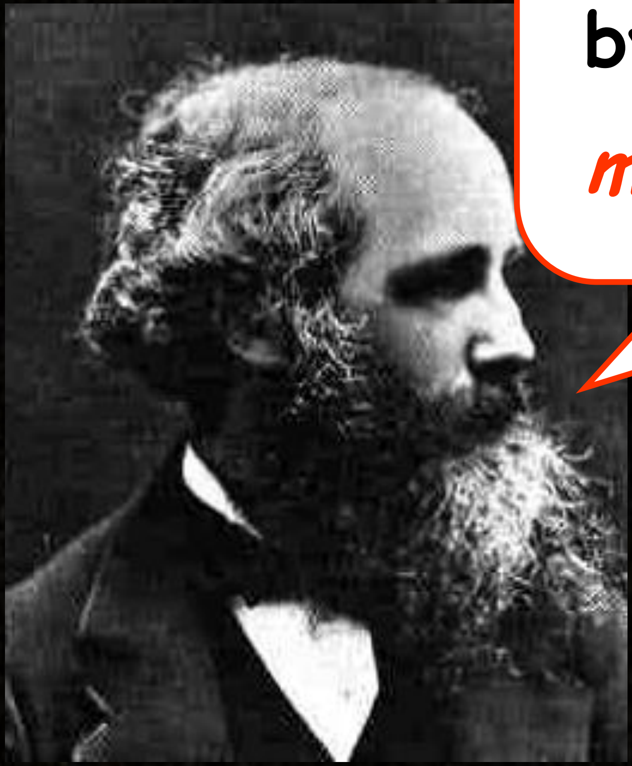
The world of Galileo and Newton (and everyday “common sense”...)

Working out how things look to different observers follows simple rules, in different *reference frames*



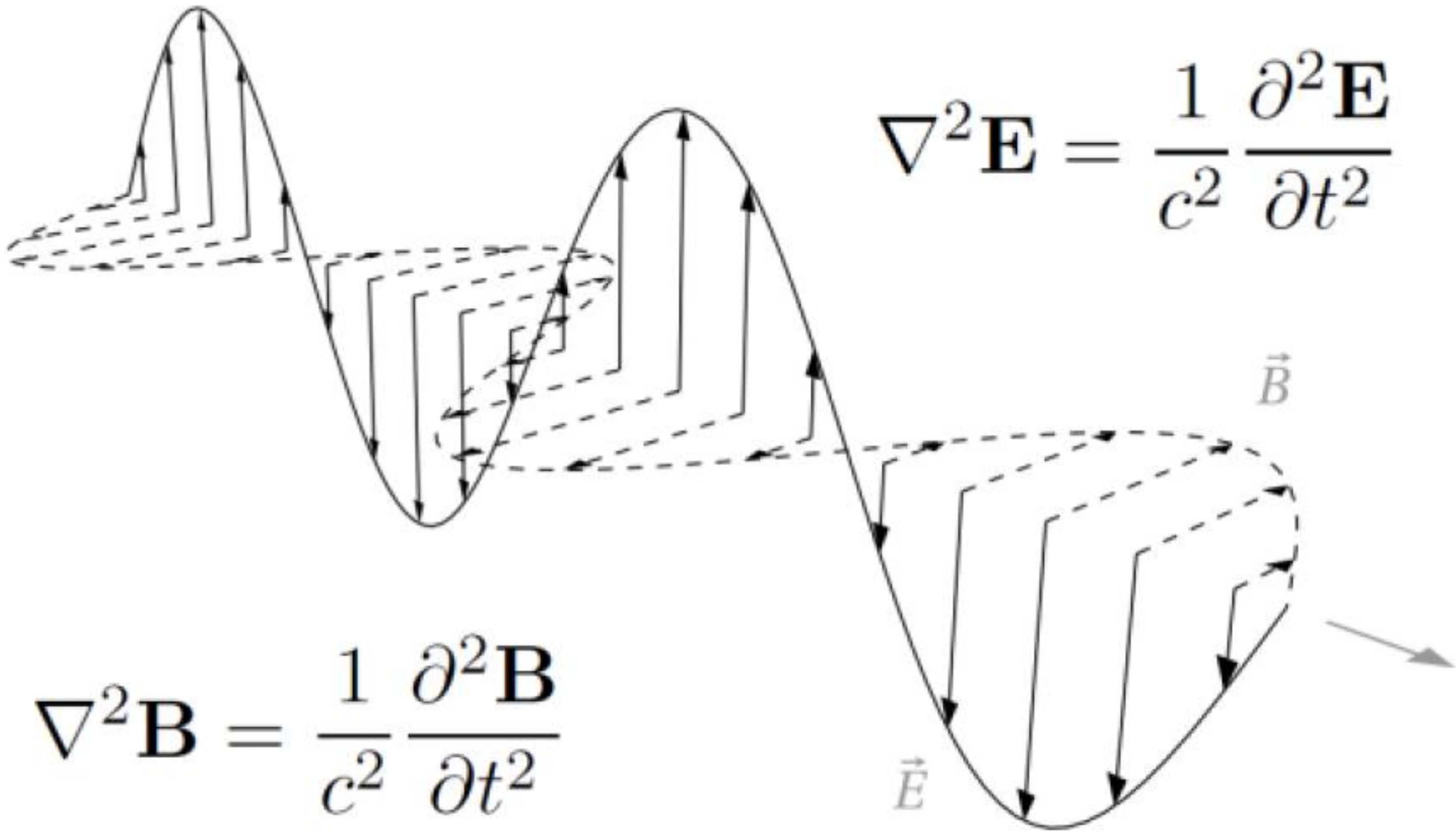
James Clerk Maxwell's theory of light

Light is a ***wave*** caused by varying ***electric*** and ***magnetic*** fields



(but how does light propagate through space?...)

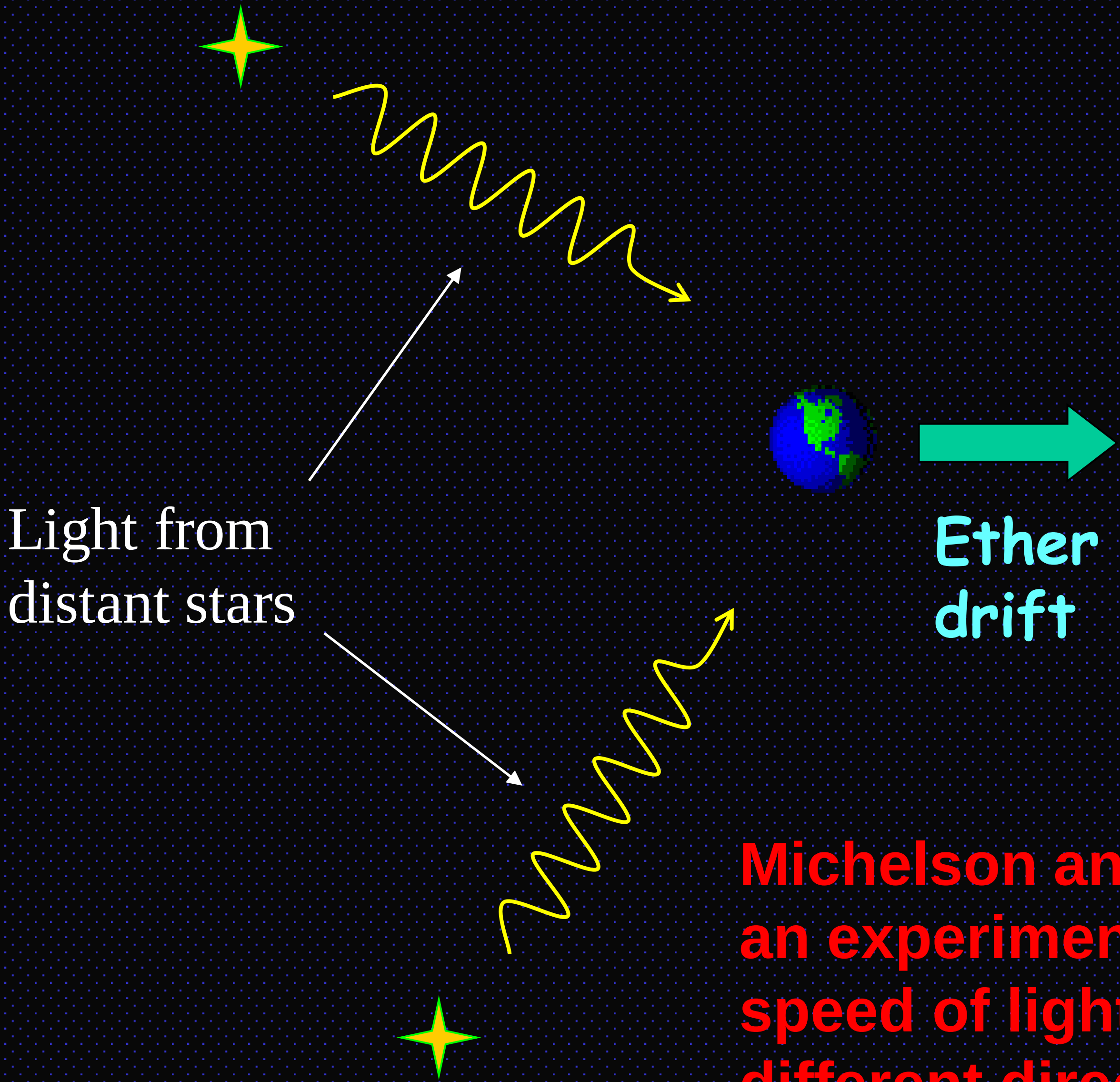
$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$



$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

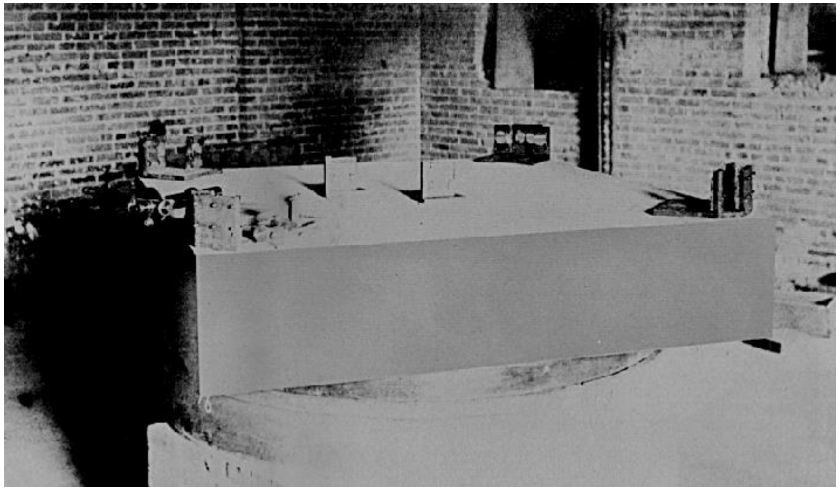
$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

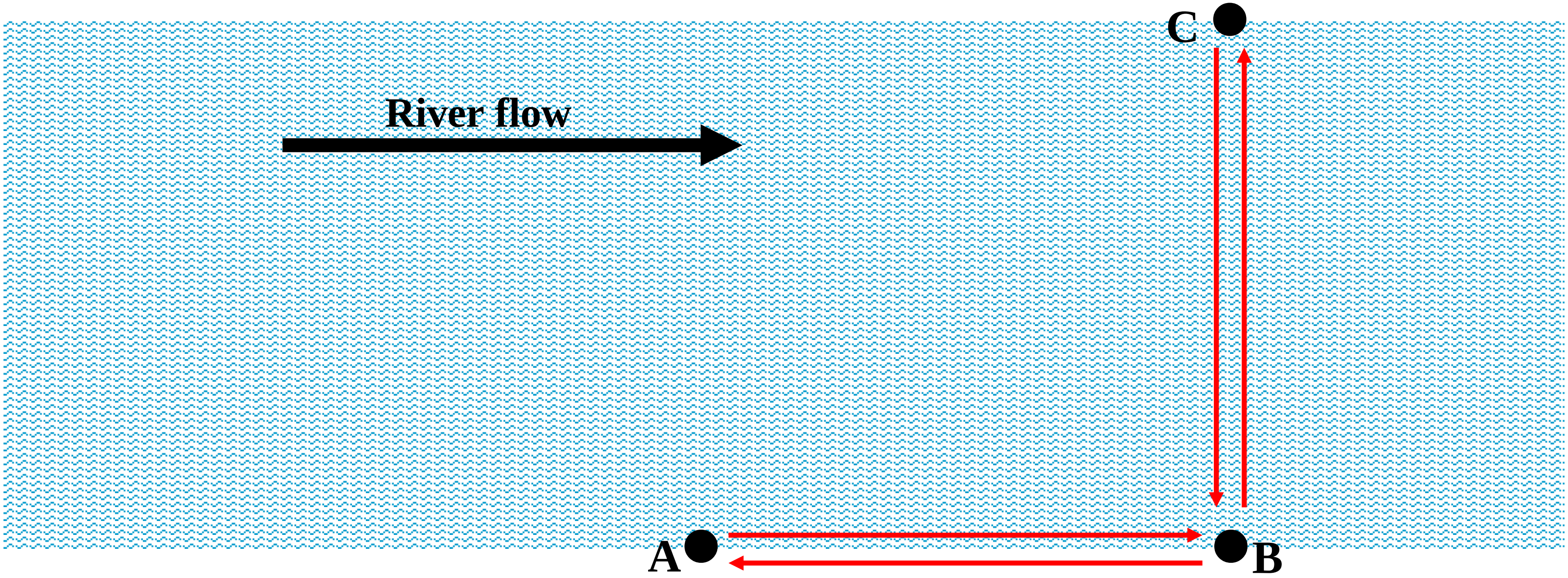
Through the Ether?…



Michelson and Morley devised an experiment to measure the speed of light coming from different directions

The Michelson and Morley Experiment would try to measure the "Ether Drift" by timing different light beams – like swimmers on a fast-flowing river





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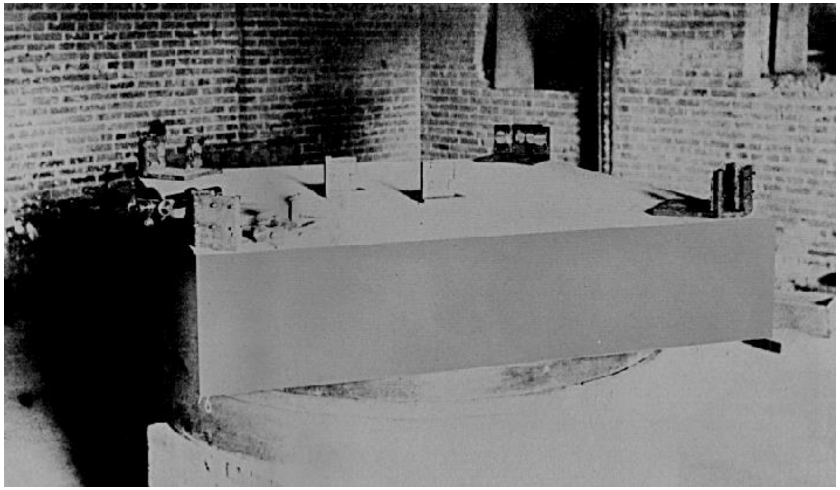
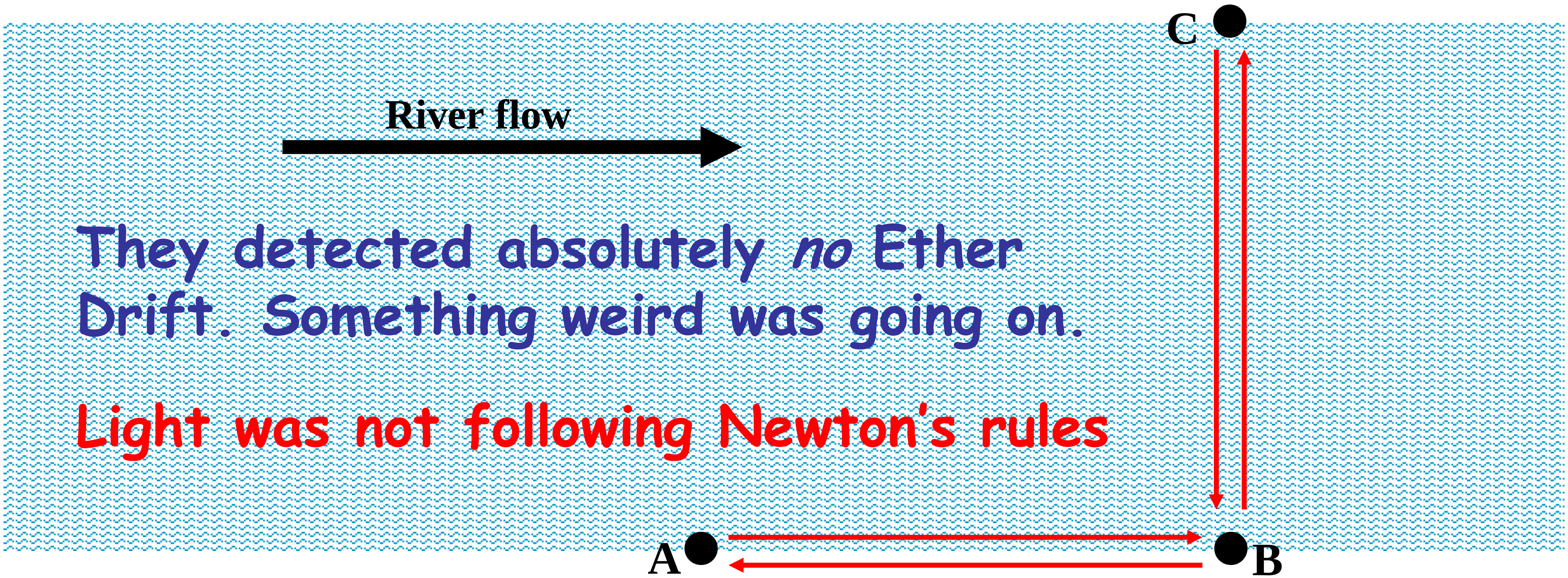
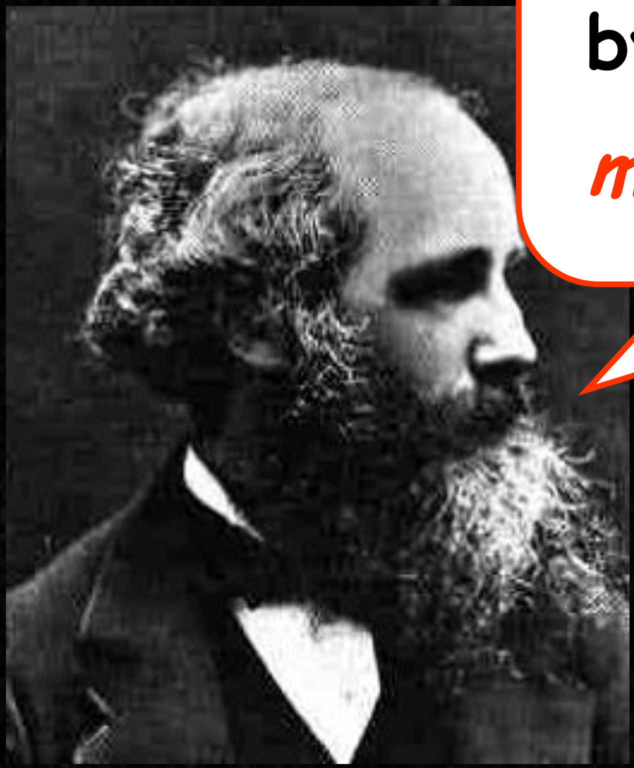




They detected absolutely *no* Ether Drift. Something weird was going on.

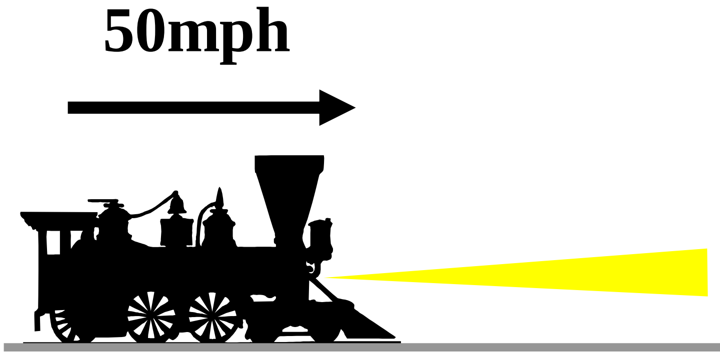
Light was not following Newton's rules

James Clerk Maxwell's theory of light

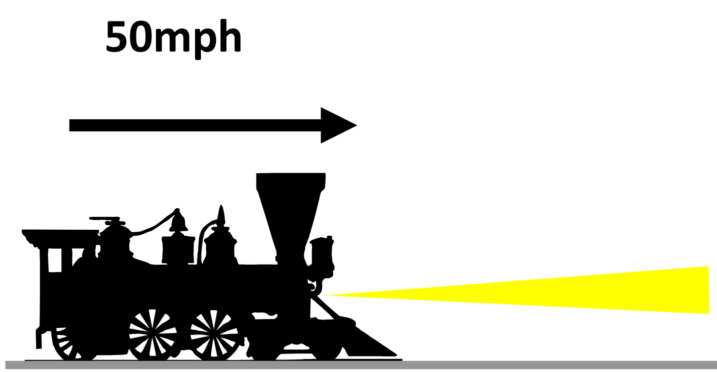


Light is a *wave* caused by varying *electric* and *magnetic* fields

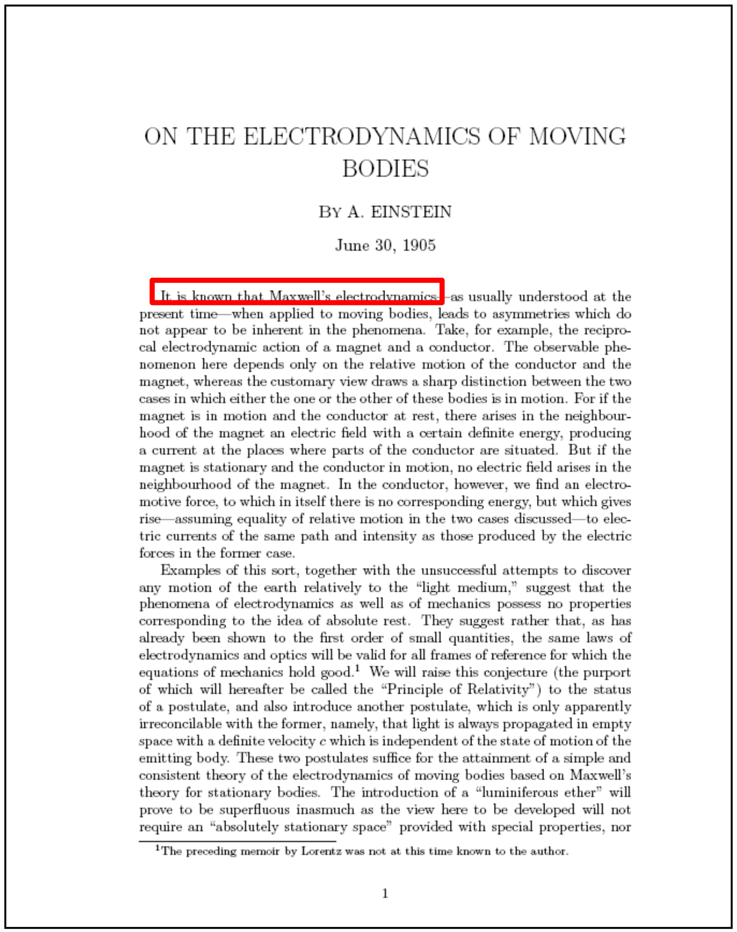
But what if I travelled *alongside* a light beam? Would it still wave?



Speed of light relative to the ground *faster* than speed of light relative to the train?



In Einstein's relativity, the speed of light is *unchanged* by the motion of the train


ON THE ELECTRODYNAMICS OF MOVING BODIES

BY A. EINSTEIN

June 30, 1905

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.[1] We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous ether" will prove to be superfluous inasmuch as the view here to be developed will not require an "absolutely stationary space" provided with special properties, nor

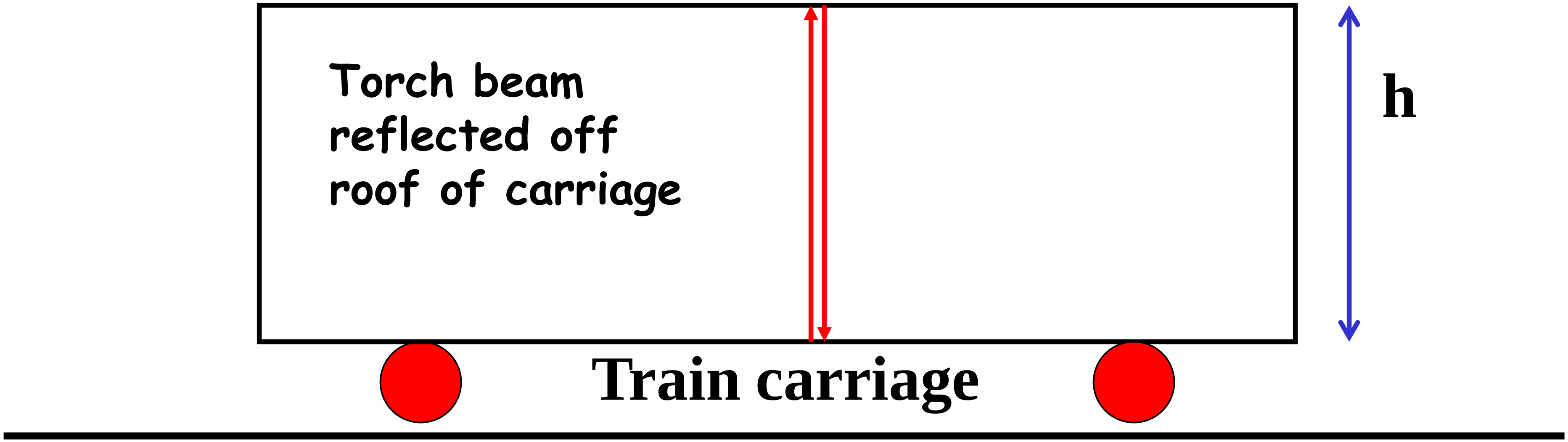
[1] The preceding memoir by Lorentz was not at this time known to the author.

1

- Measurements of space and time are ***relative*** and depend on our motion
- Unified ***spacetime*** - only measurements of the ***spacetime*** interval are invariant
- Equivalence of **matter** and **energy**

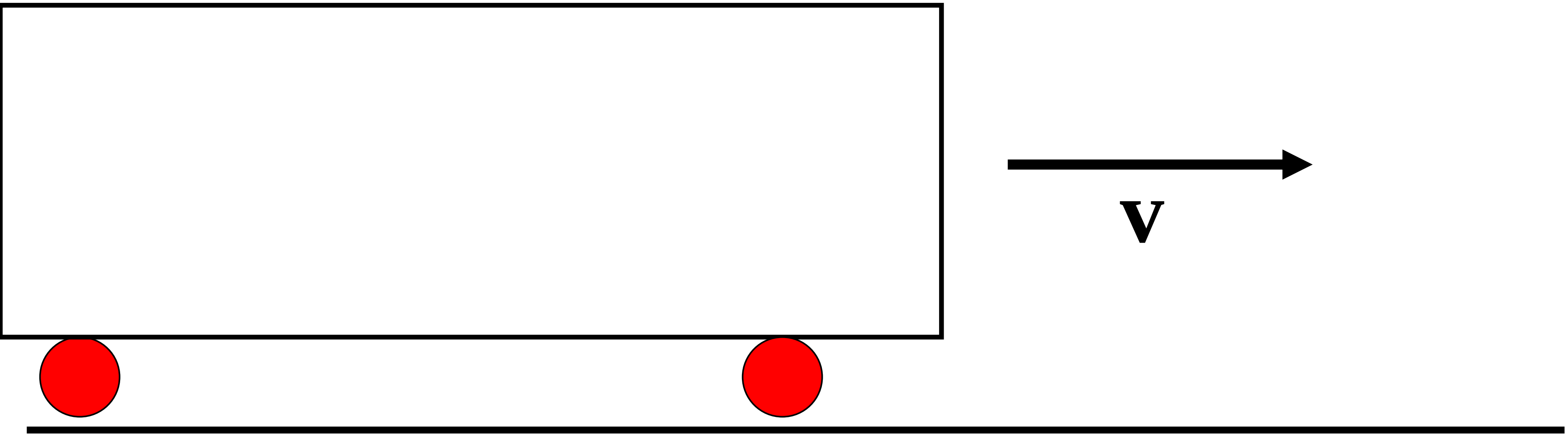
$$E = mc^2$$

Distance = speed x time



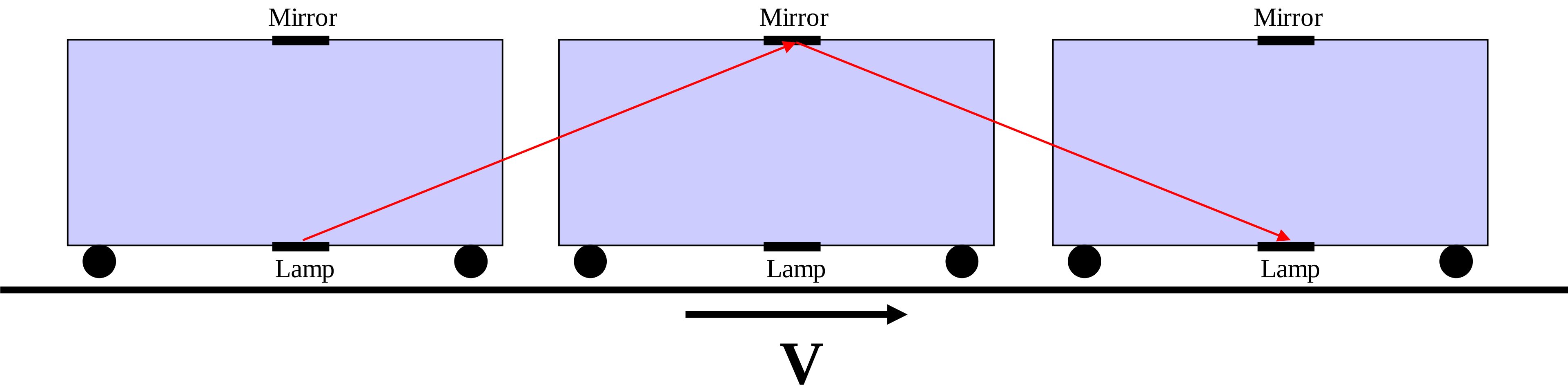
$$2h = c \times t_c$$

Now viewed from the platform…

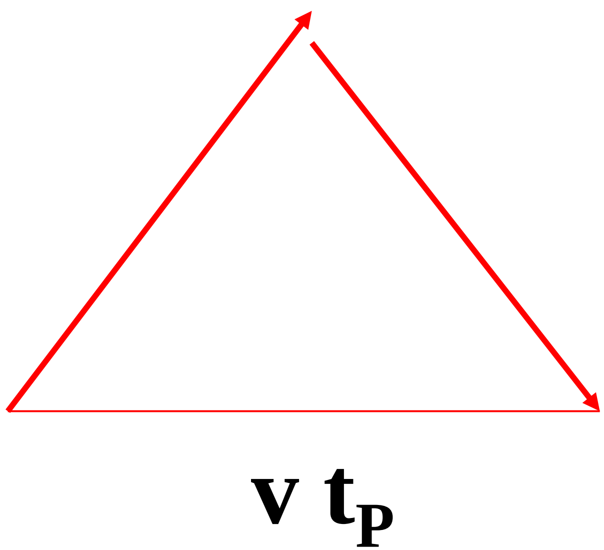


As seen by an observer on the station platform…

Light beam appears to follow a longer, diagonal path due to the motion of the train through the station.

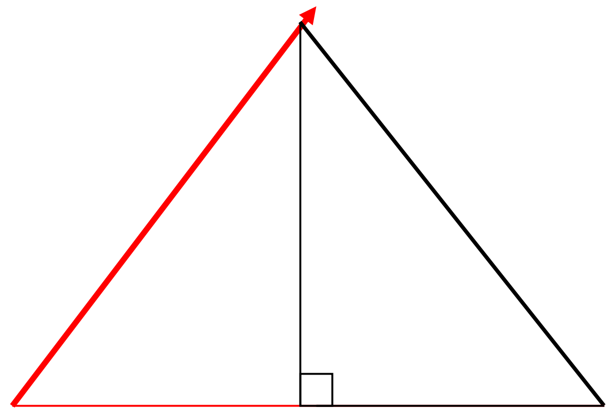


Now viewed from the platform…



The base of this triangle is $v\, t_P$

Now viewed from the platform…



This is an isosceles triangle, so it's made up of two equal right angled triangles

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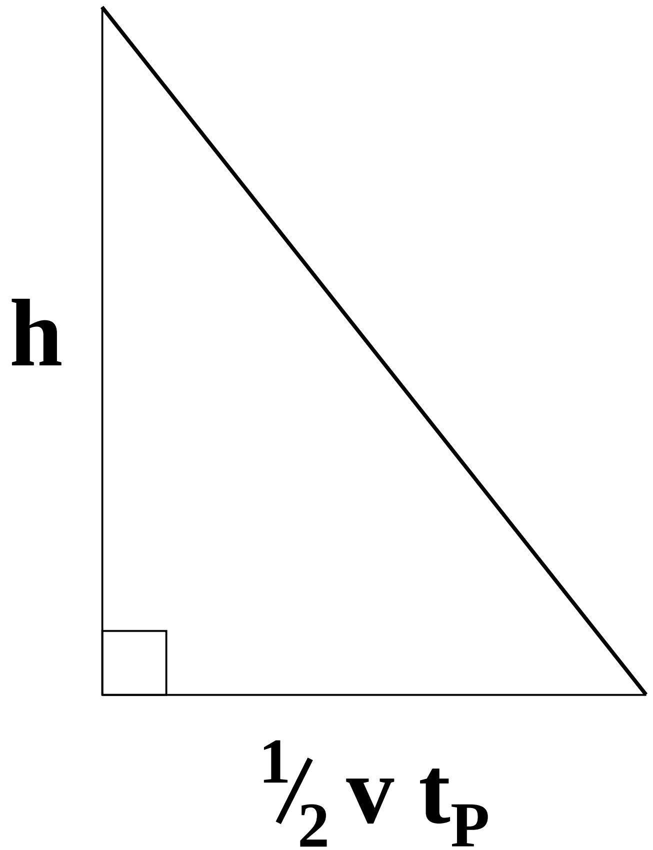
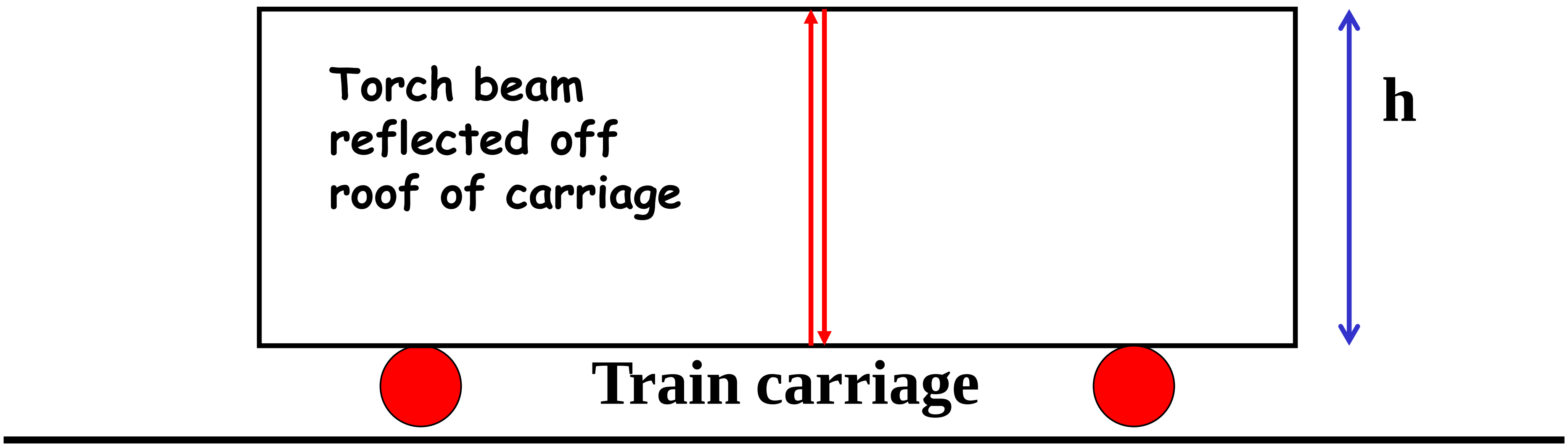


Remember: $$2h = c \times t_c$$

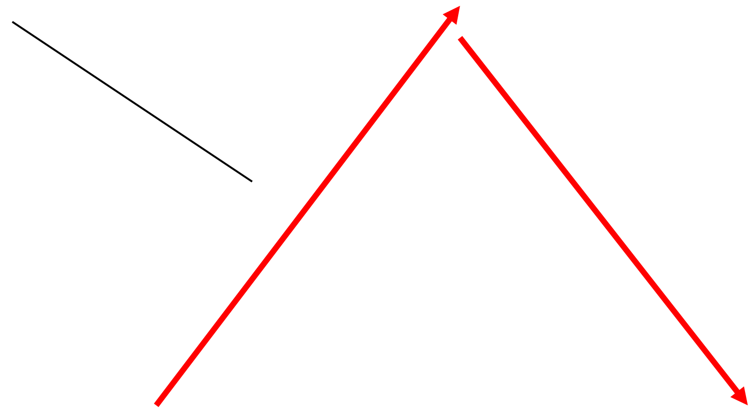
Distance = speed x time



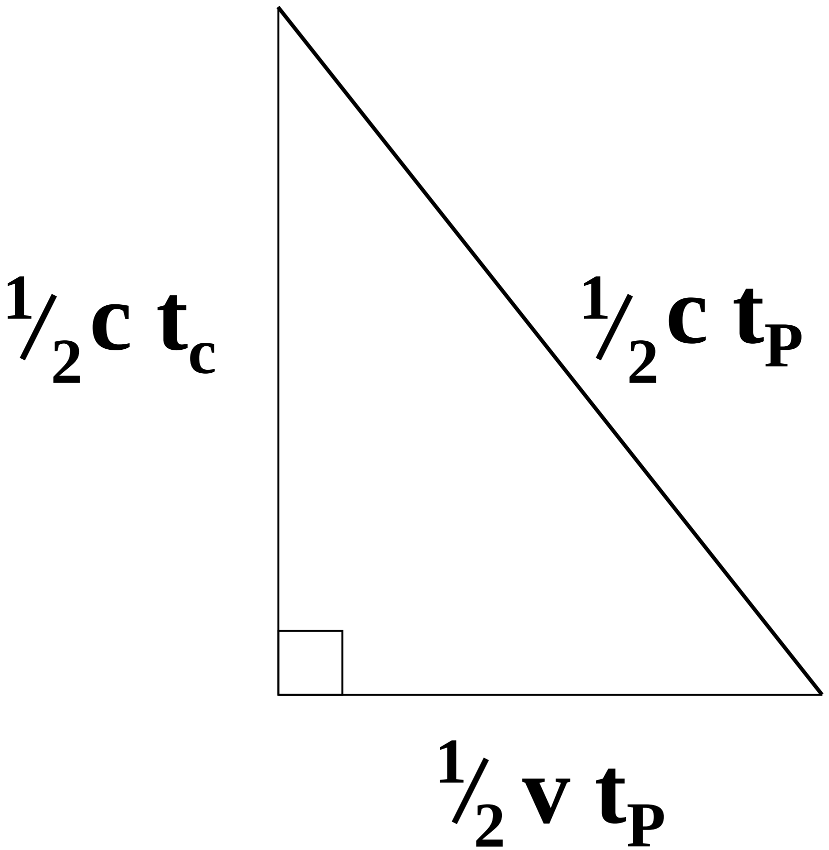
$$2h = c \times t_c$$

Total distance

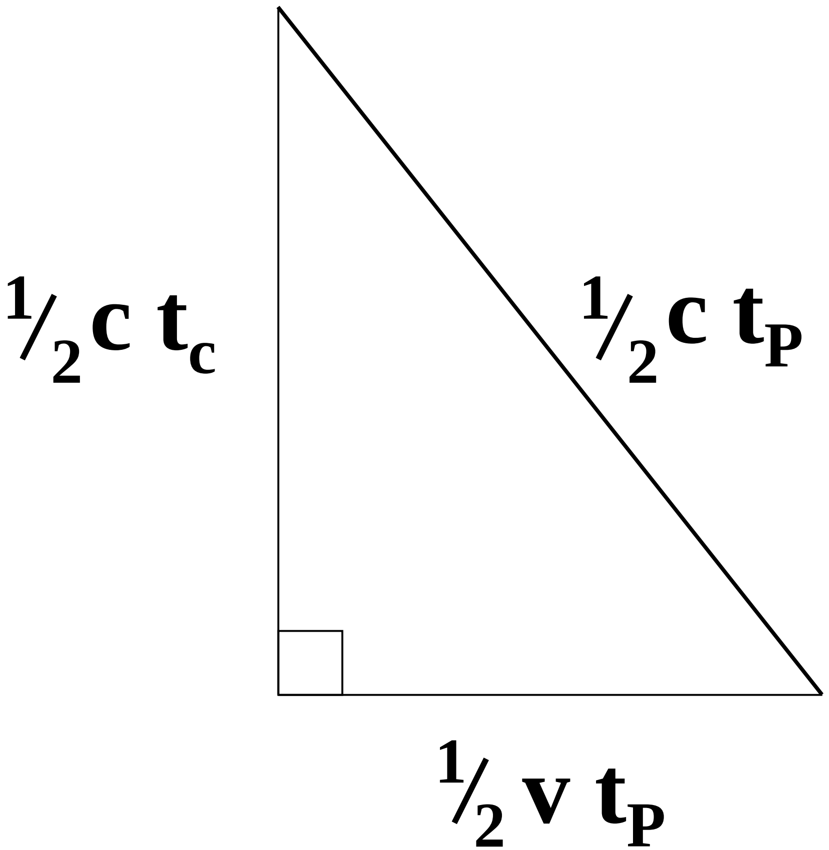
$= c \times t_P$



If **both observers measure the same speed of light, c…**



If both observers measure the same speed of light, c…



$$t_c = t_P \sqrt{(1 - v^2/c^2)}$$

To the observer on the platform, it appears that time is running more slowly on the moving train!!

$$t_c = t_P \sqrt{(1 - v^2/c^2)}$$

or

$$t_P = \frac{t_c}{\sqrt{(1 - v^2/c^2)}}$$

Q: ***But why can't I think of the train as stationary, and the platform that's rushing past?...***

Wouldn't that mean that, to an observer onboard the train it's the clock on the platform that appears to be running slowly?

A: ***YES!!!*** ***(and this feels weird....)***

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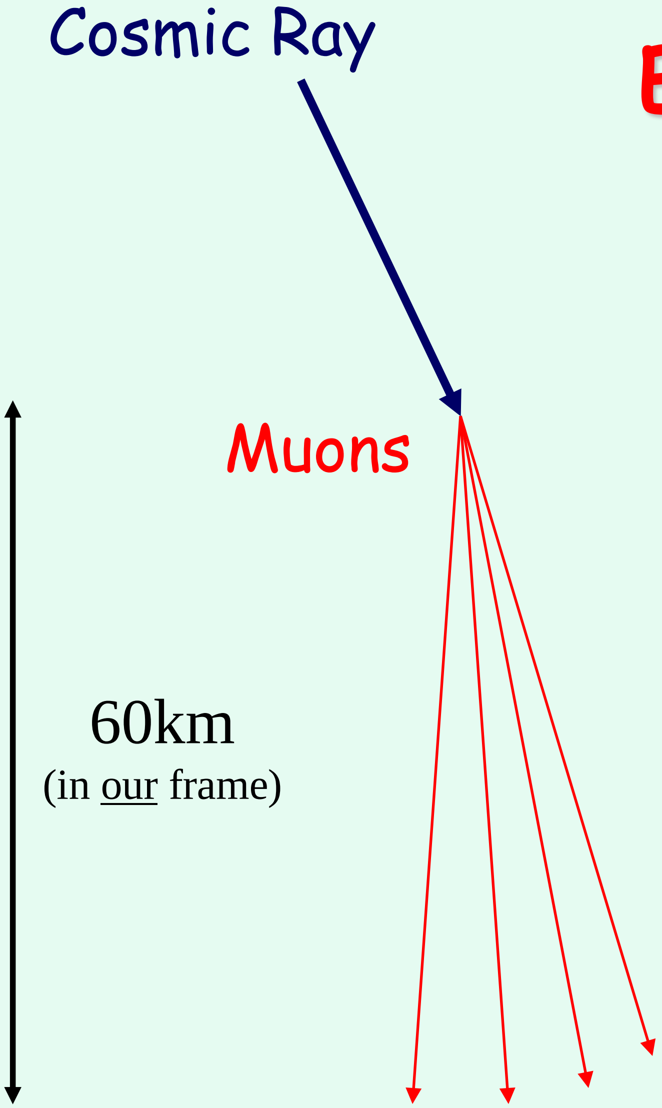
Evidence for Time Dilation

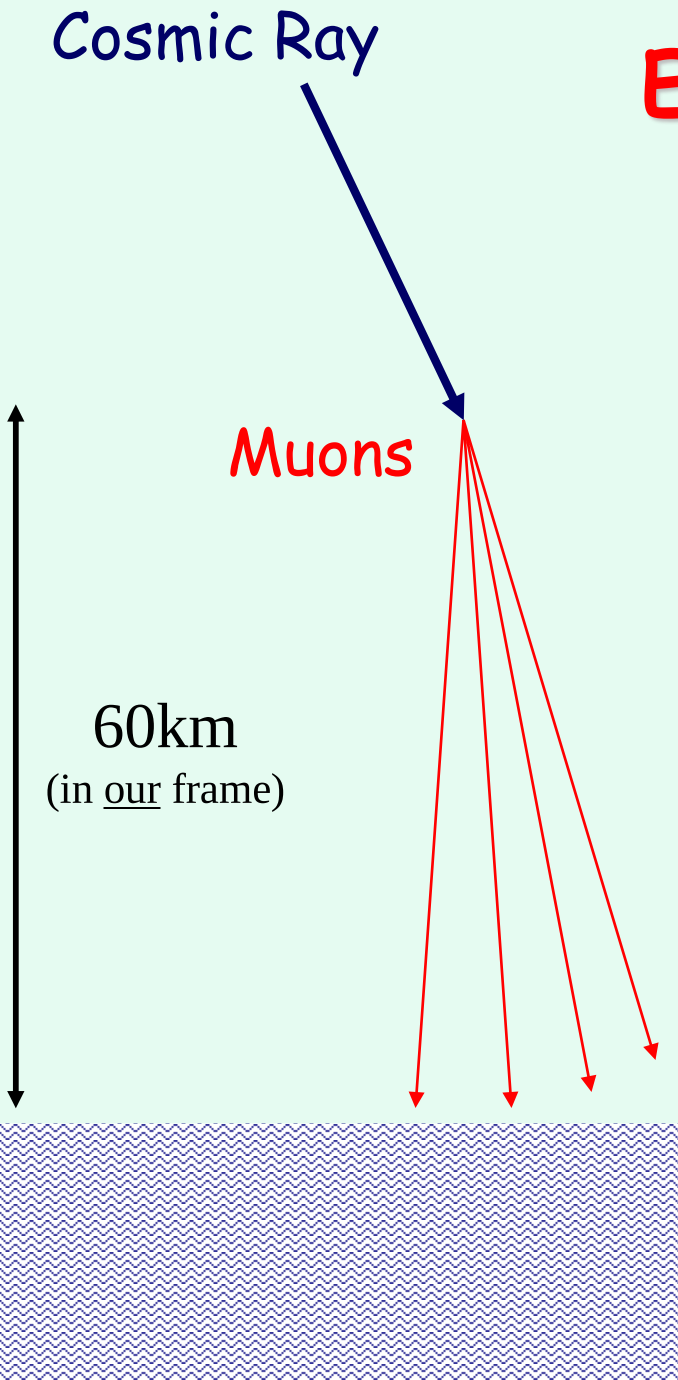


Slow moving muons would never reach sea level, because their half-life is only 2.2 micro-seconds

But v = 0.999c, so muon lifetime appears to us to be greatly extended

Sea level

Evidence for Time Dilation



In the muons' frame their half-life is *still* only 2.2 μs

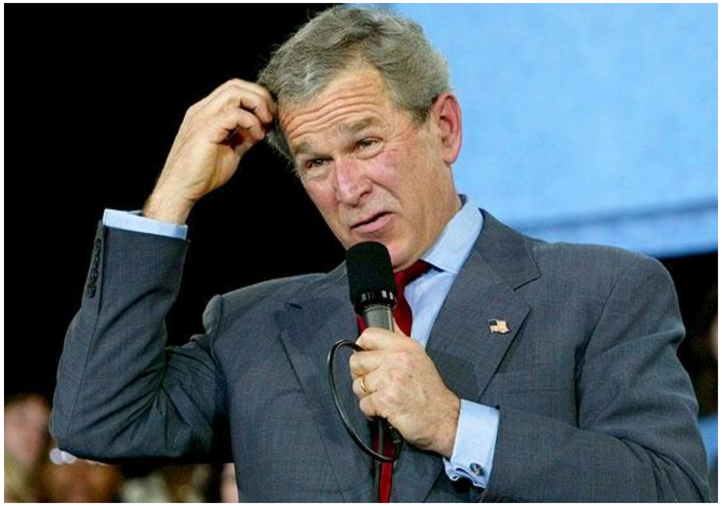
So, from the muons' point of view it's the *distance* they travel which is much shorter

Length Contraction

There is much scope for head-scratching – and at first glance what look like **paradoxes** – when we try to switch reference frames.

1. Pole in the barn paradox

2. Twins paradox



Resolution is to do with what we mean by **simultaneity** and the **interval** between spacetime events.

1. Foundations of General Relativity (pgs. 6 – 12)

GR is a generalisation of Special Relativity (1905).

In SR Einstein formulated the laws of physics to be valid for all **inertial observers**

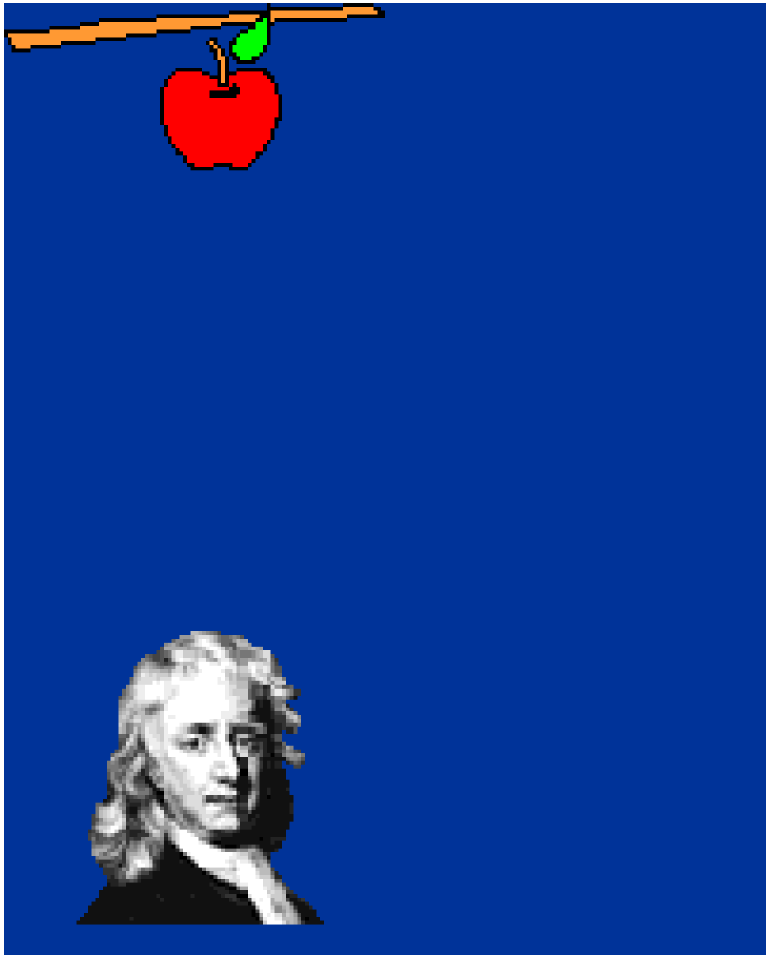
→ **Measurements of space and time relative to observer's motion.**

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Minkowski metric

Invariant interval

Newtonian gravity is incompatible with SR

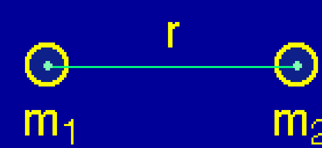


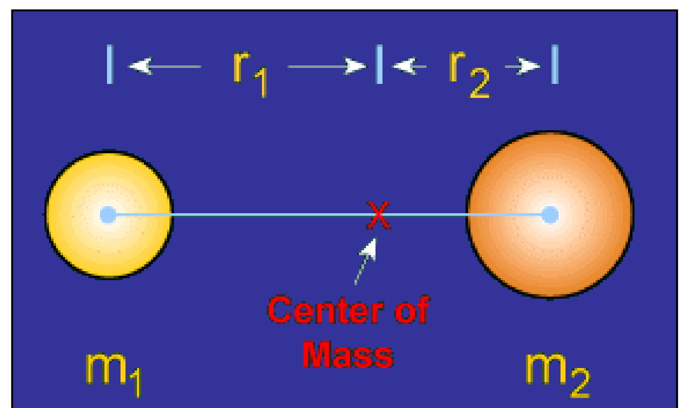
Isaac Newton:
1642 – 1727 AD

Law of Universal Gravitation

Every object in the Universe attracts every other object with a force directed along the line of centers for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation between the two objects.

$$F_g = G\frac{m_1 m_2}{r^2}$$





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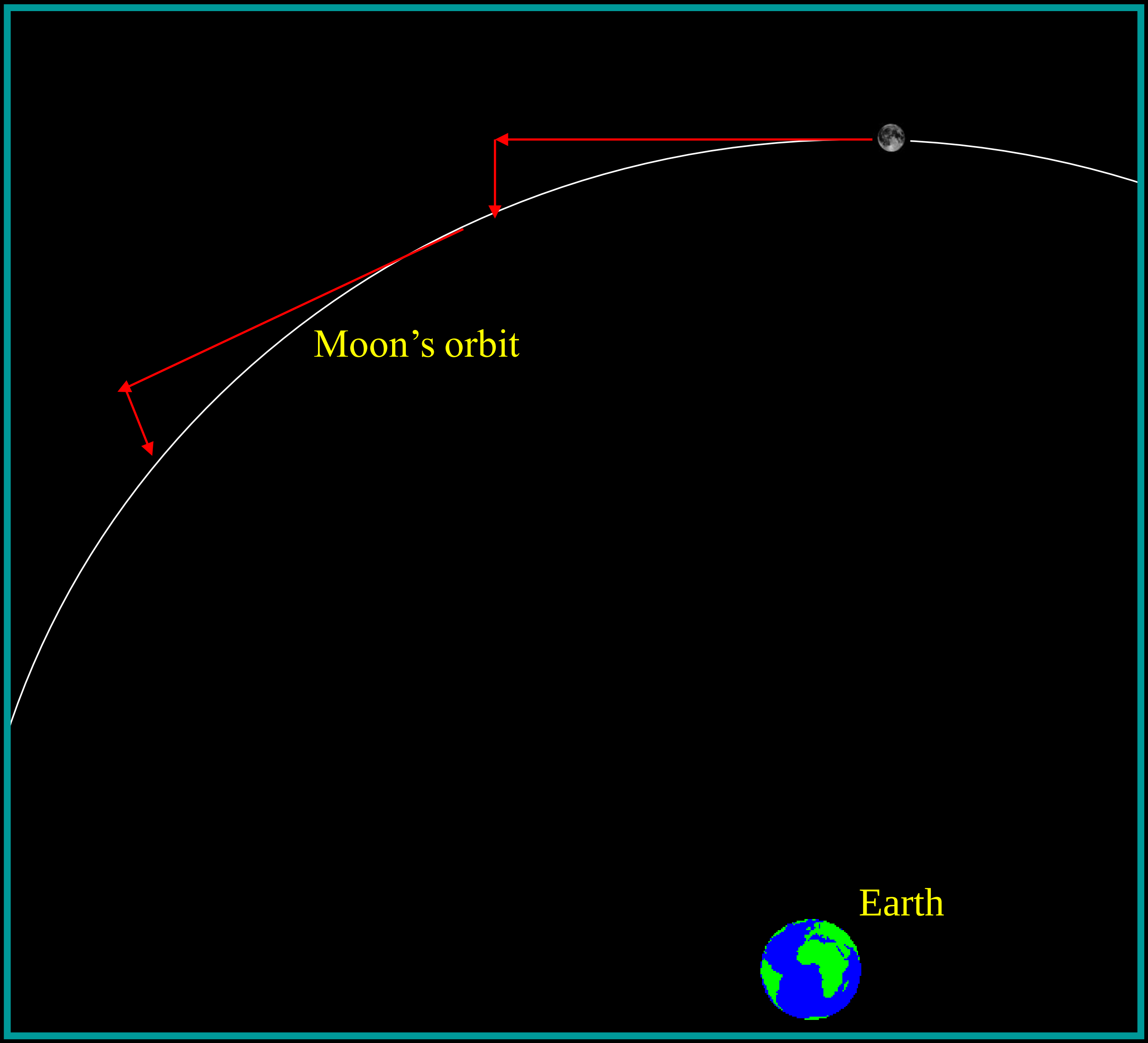
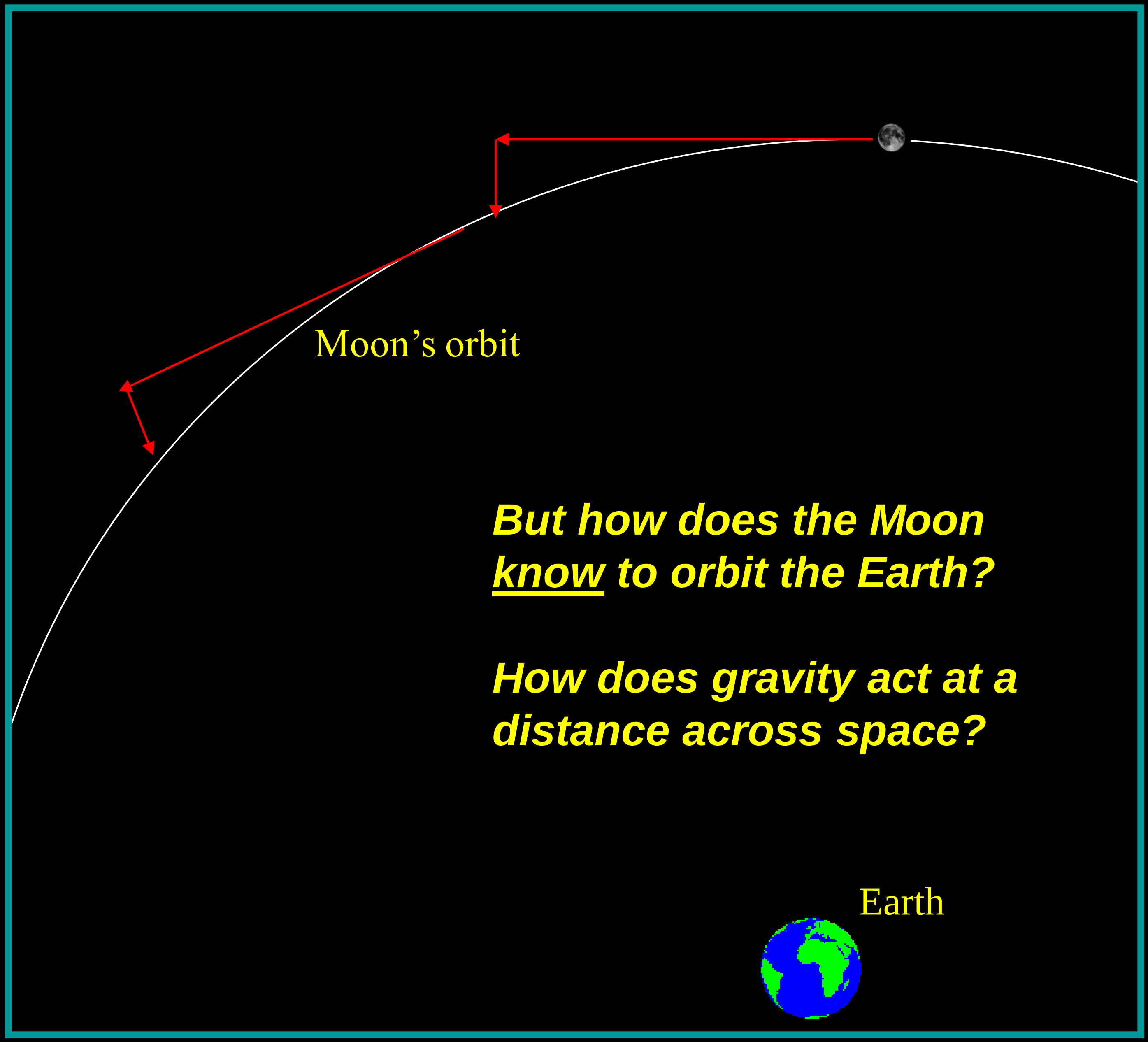

Moon’s orbit
Earth


Moon's orbit
But how does the Moon know to orbit the Earth?
How does gravity act at a distance across space?
Earth

Principles of Equivalence

Inertial Mass $\vec{F}_I = m_I \vec{a}$

Gravitational Mass $\vec{F}_G = \frac{m_G M}{r^2} \hat{r} \equiv m_G \vec{g}$

Weak Equivalence Principle

$$m_I = m_G$$

Gravity and acceleration are ***equivalent***

The WEP implies:

A object freely-falling in a uniform gravitational field inhabits an **inertial frame** in which all gravitational forces have disappeared.

*But only **LIF**: only local over region for which gravitational field is uniform.*

Strong Equivalence Principle

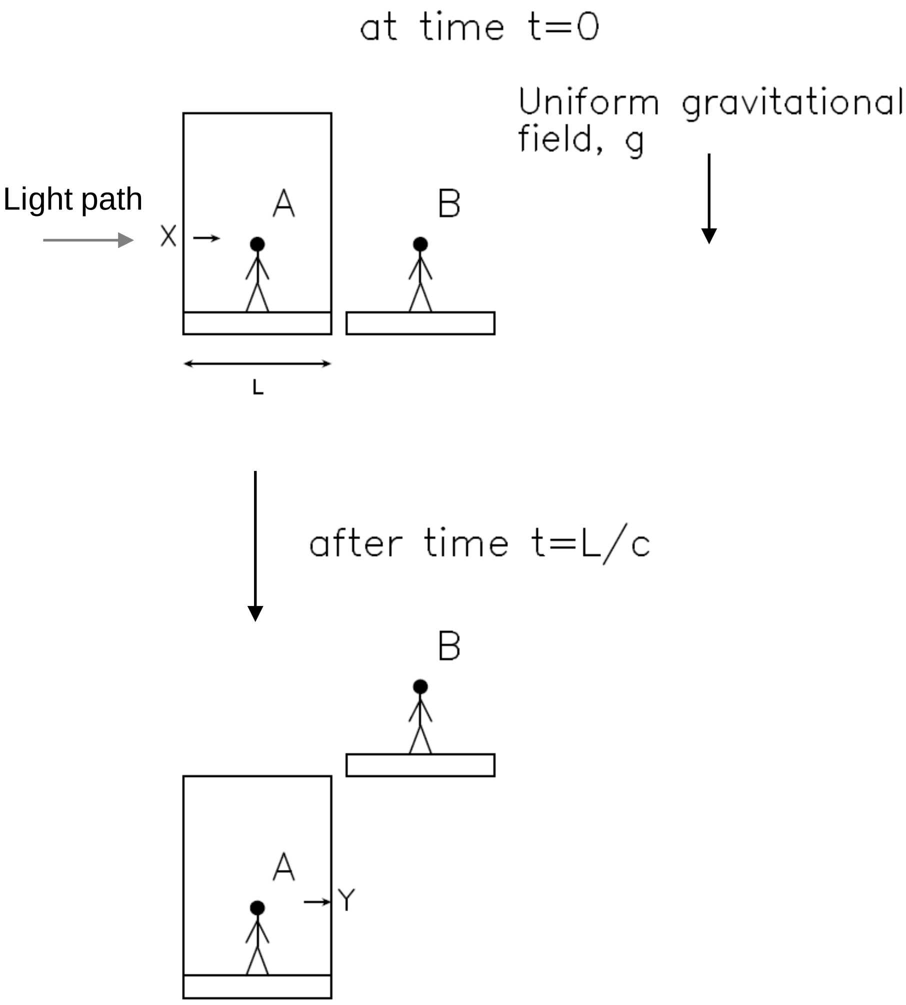
Locally (i.e. in a LIF) *all* laws of physics reduce to their SR form – apart from gravity, which simply disappears.

The Equivalence principles also predict gravitational light deflection…

Light enters lift horizontally at X, at instant when lift begins to free-fall.

Observer A is in LIF. Sees light reach opposite wall at Y (same height as X), in agreement with SR.

To be consistent, observer B outside lift must see light path as ***curved****, interpreting this as due to the gravitational field*

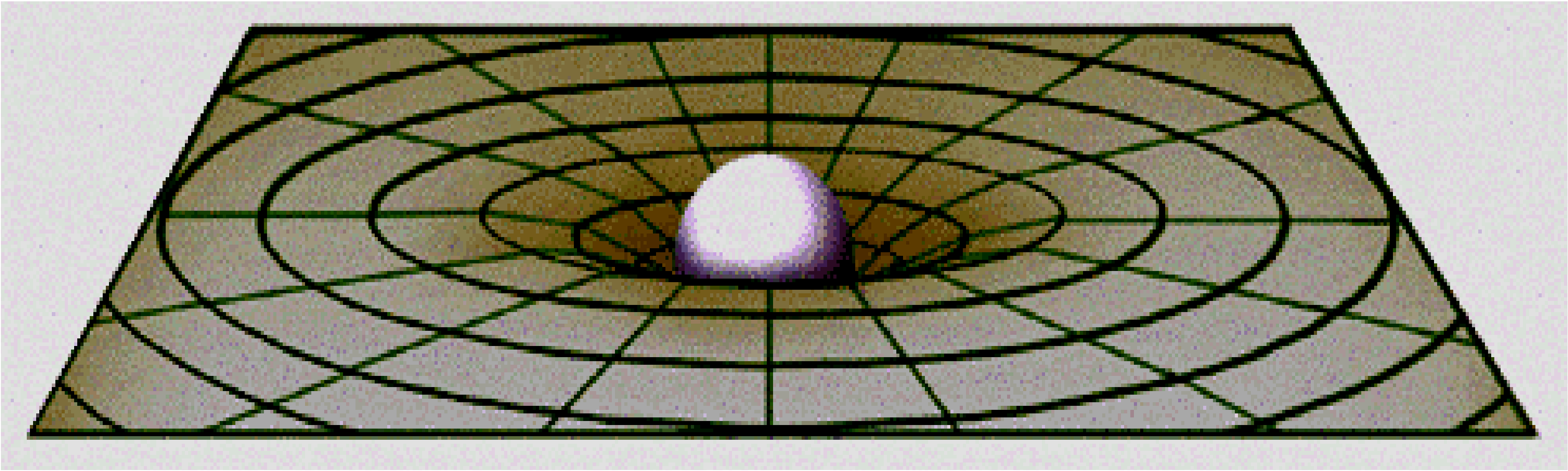


Similarly, EPs predict gravitational redshift

2. Introduction to Geodesic Deviation (pgs.13 – 17)

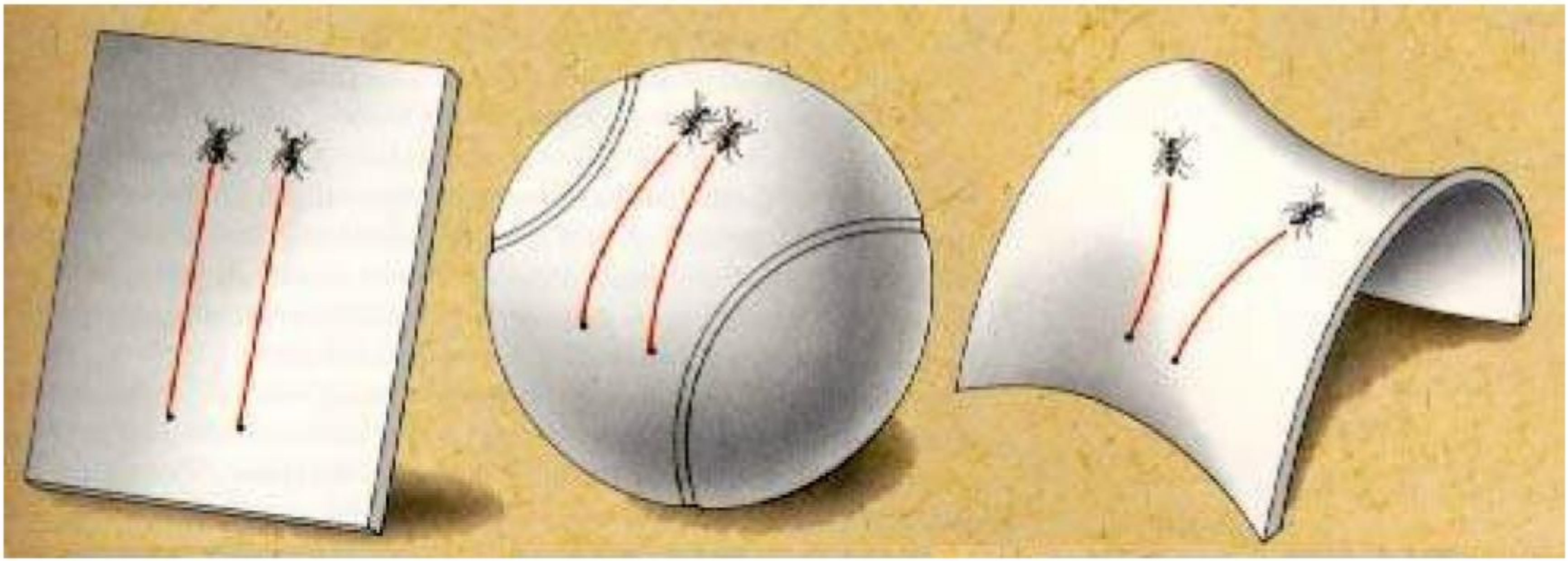
In GR trajectories of freely-falling particles are **geodesics** – the equivalent of straight lines in curved spacetime.

Analogue of Newton I: Unless acted upon by a non-gravitational force, a particle will follow a geodesic.



The curvature of spacetime is revealed by the behaviour of neighbouring geodesics.

Consider a 2-dimensional analogy.



Zero curvature: geodesic deviation unchanged.

Positive curvature: geodesics converge

Negative curvature: geodesics diverge

Non-zero curvature

$$\Leftrightarrow$$

Acceleration of geodesic deviation

$$\Leftrightarrow$$

Non-uniform gravitational field

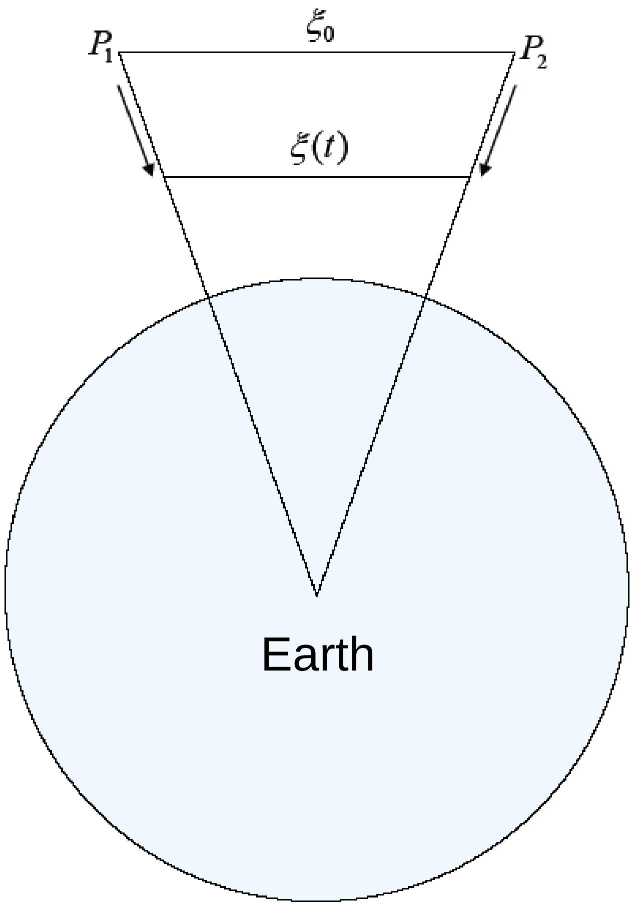
We can first think about geodesic deviation and curvature in a Newtonian context

By similar triangles

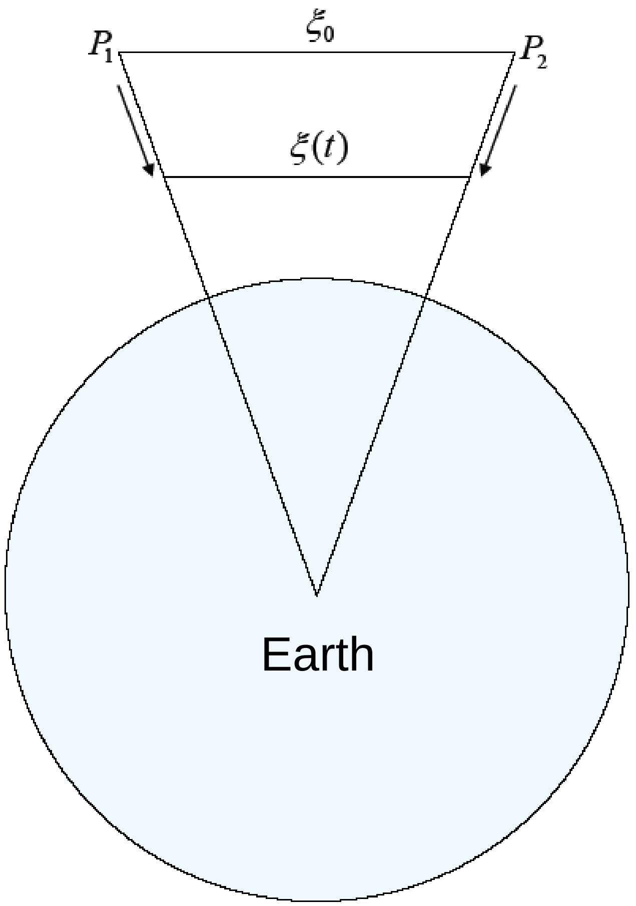
$$\frac{\xi(t)}{r(t)} = \frac{\xi_0}{r_0} = k$$

Hence

$$\ddot{\xi} = k\ddot{r} = -\frac{kGM}{r^2}$$



We can first think about geodesic deviation and curvature in a Newtonian context



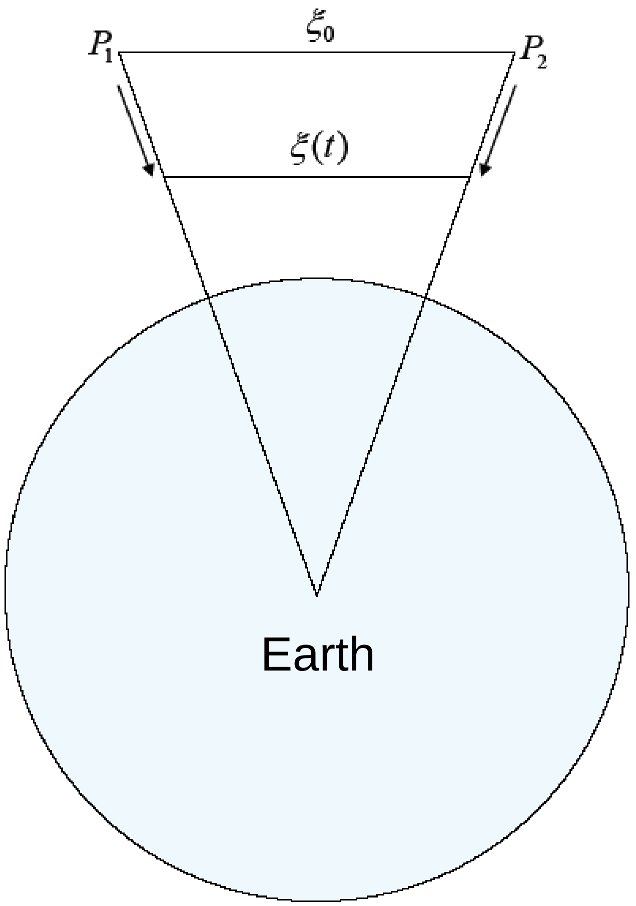
or

$$\ddot{\xi} = -\frac{\xi}{r}\frac{GM}{r^2} = -\frac{GM\xi}{r^3}$$

which we can re-write as

$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM}{R^3c^2}\,\xi$$

We can first think about geodesic deviation and curvature in a Newtonian context



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which we can re-write as

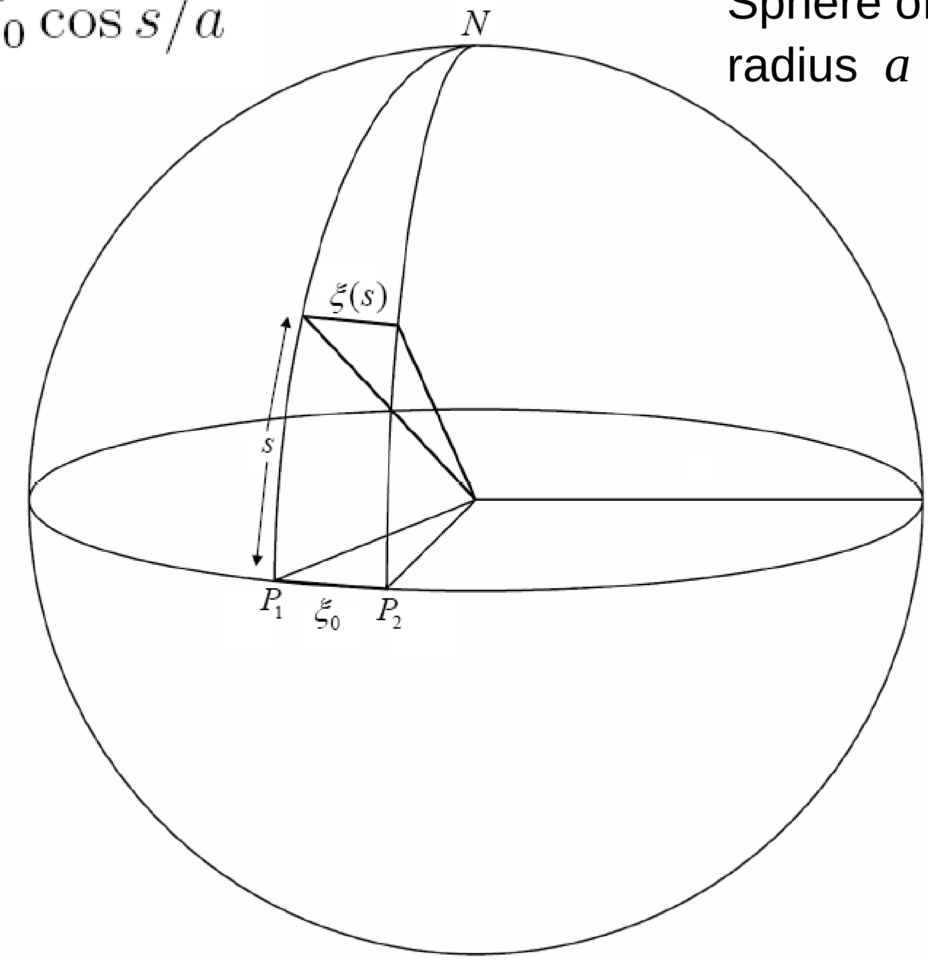
$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM}{R^3c^2}\,\xi$$

At Earth's surface this equals $2\times10^{-23}\ \mathrm{m}^{-2}$

Another analogy will help us to interpret this last term

$$\xi(s) = a\cos\theta\, d\phi = \xi_0 \cos\theta = \xi_0 \cos s/a$$

Sphere of radius a



Differentiating: $$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi$$

Another analogy will help us to interpret this last term

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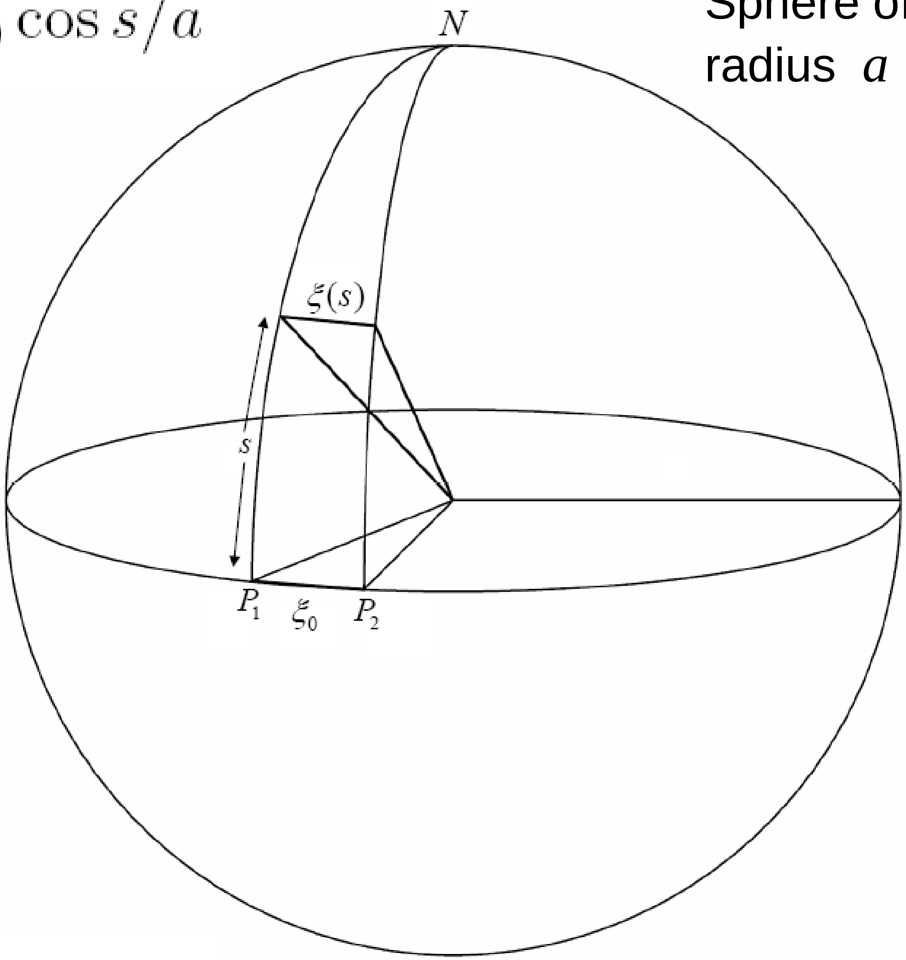


Differentiating: $$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi$$

Comparing with previous slide:

$$\mathcal{R} = \left\{ \frac{GM}{R^3 c^2} \right\}^{-\frac{1}{2}}$$

represents radius of curvature of spacetime at the Earth's surface

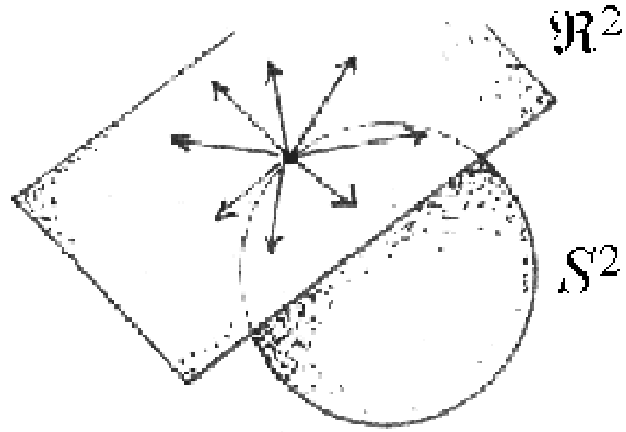
$$\mathcal{R} \sim 2 \times 10^{11} \text{ m}$$

3. A Mathematical Toolbox for GR (pgs.18 – 32)

Riemannian Manifold

A continuous, differentiable space which is locally **flat** and on which a distance, or **metric**, function is defined.

(e.g. the surface of a sphere)



The tangent space in a generic point of an $\mathbb{S}^2$ sphere

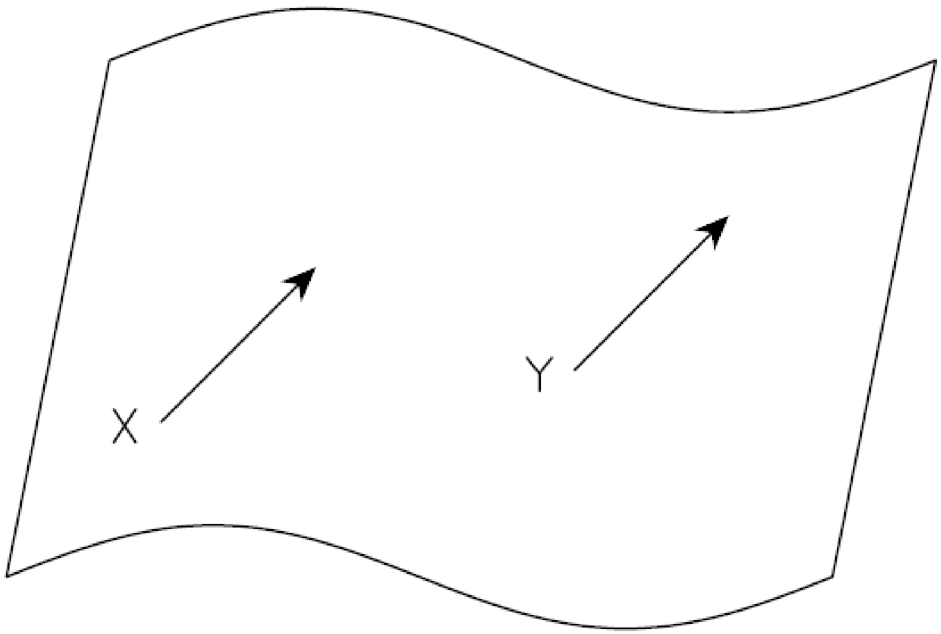
The mathematical properties of a Riemannian manifold match the physical assumptions of the strong equivalence principle

Vectors on a curved manifold

We think of a vector as an arrow representing a displacement.

$$\Delta\vec{x} = \Delta x^{\alpha}\,\vec{e}_{\alpha}$$

components — basis vectors

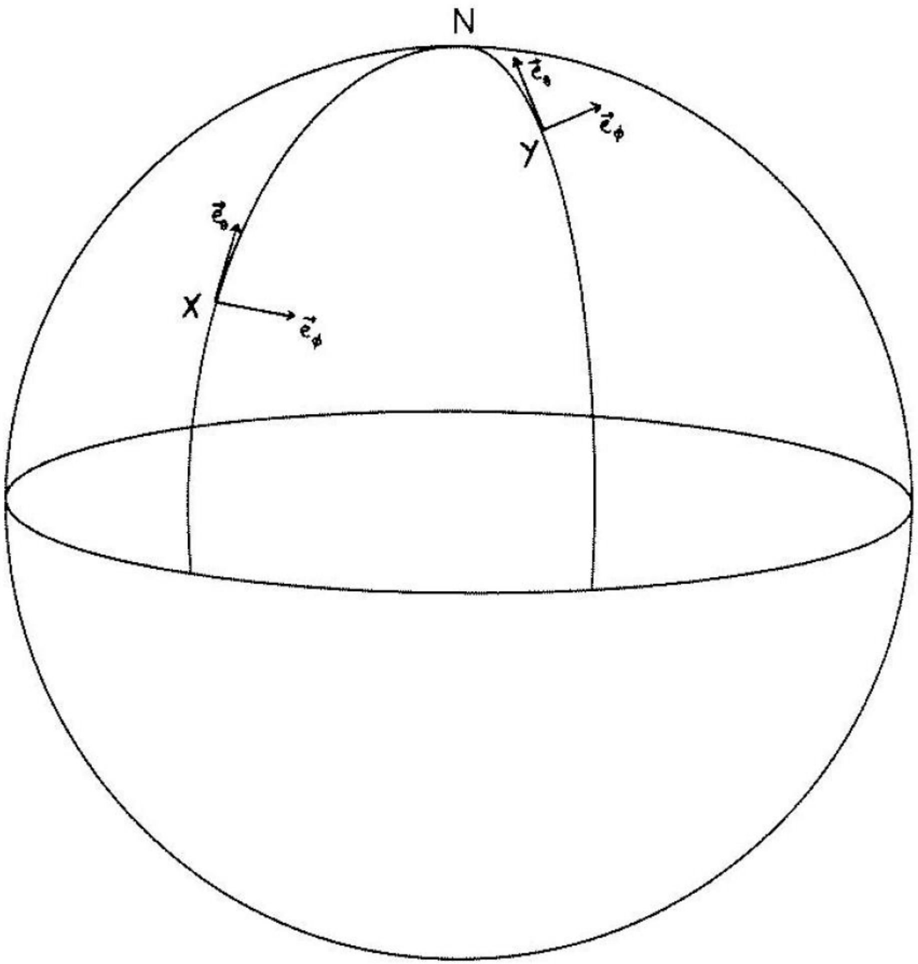


In general, components of vector different at X and Y, even if the vector is the same at both points, because the basis vectors change

Simple example: 2-D sphere.

Set of curves parametrised by ***coordinates***

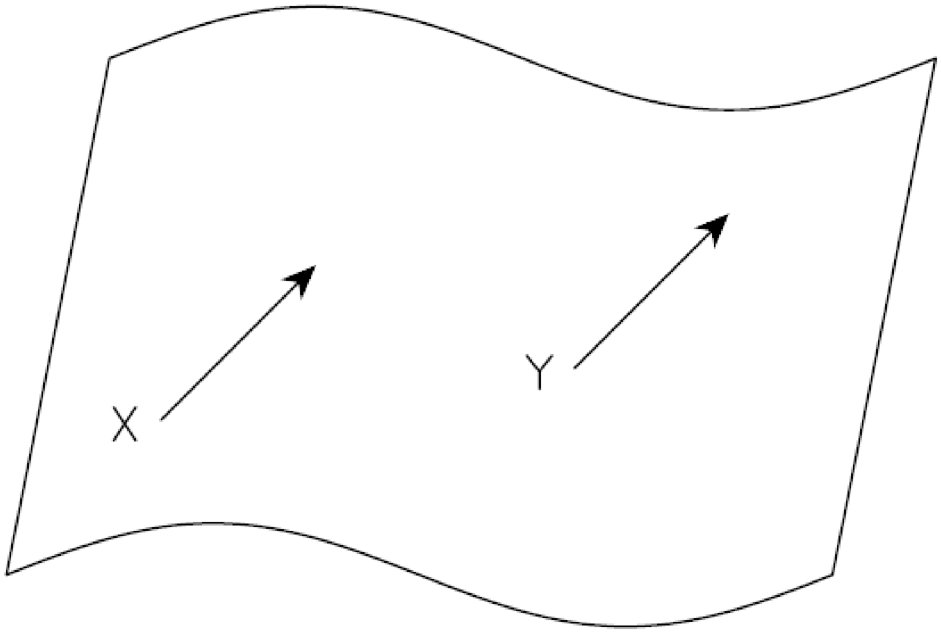
$$\vec{e}_i \equiv \frac{\partial}{\partial x^i}$$ ***tangent*** to i^{th} curve



Basis vectors different at X and Y.

We need to be able to handle changing basis vectors and components on surfaces of arbitrary curvature.

We need rules to tell us how to express the components of a vector in a different coordinate system, and at different points in our manifold.



e.g. in new, dashed, coordinate system, by the chain rule

$$\Delta x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha}$$

This is a **transformation law**.

Definition of a vector: any set of components that transforms like this

Generalisation to tensors

We can construct more general geometrical objects called **tensors** which again are defined by how they transform.

e.g. an *(l,m)* tensor is a linear operator with transformation law

$$A'^{u_1\,u_2\,\ldots\,u_l}_{r_1\,r_2\,\ldots\,r_m} = \frac{\partial x'^{u_1}}{\partial x^{t_1}} \ldots \frac{\partial x'^{u_l}}{\partial x^{t_l}} \frac{\partial x^{q_1}}{\partial x'^{r_1}} \ldots \frac{\partial x^{q_m}}{\partial x'^{r_m}} A^{t_1\,t_2\,\ldots\,t_l}_{q_1\,q_2\,\ldots\,q_m}$$

Vectors are the special case of a *(1,0)* tensor.

If a tensor equation can be shown to be valid in a particular coordinate system, it must be valid in *any* coordinate system.

Example:

metric tensor

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Remember, for SR

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Minkowski metric

Invariant interval

These can be functions of space or time

Now generalises to

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Invariant interval (scalar)

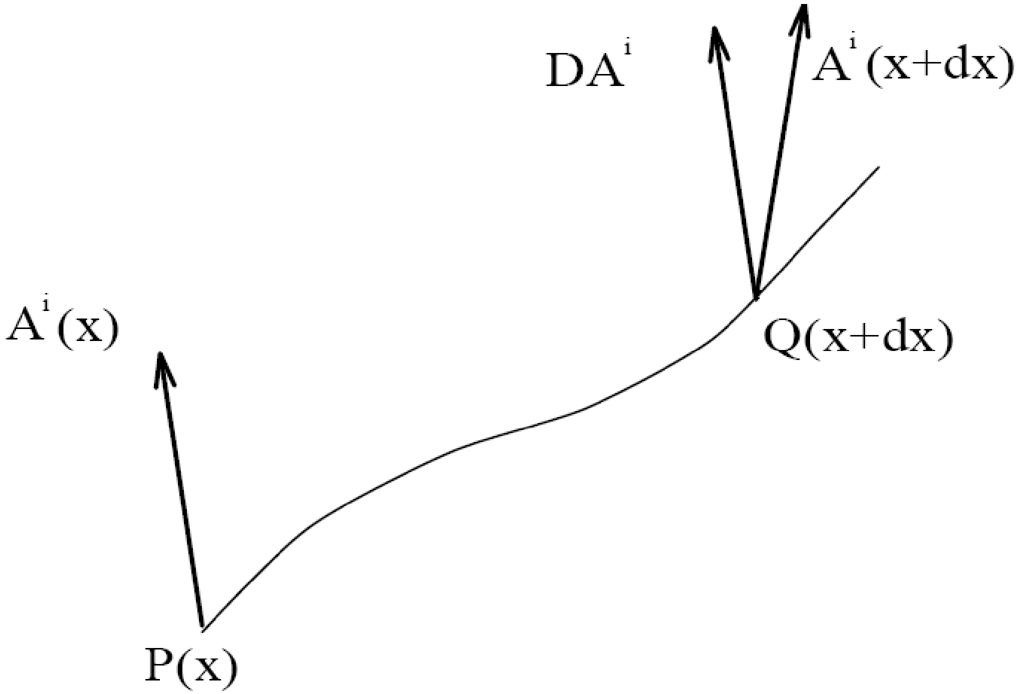
Contravariant vectors or (1,0) tensors

Covariant differentiation

In physics we need to be able to differentiate quantities like vectors – this involves subtracting components at neighbouring points.

This is a problem because the transformation law for the components of vectors will in general be different at P and Q.

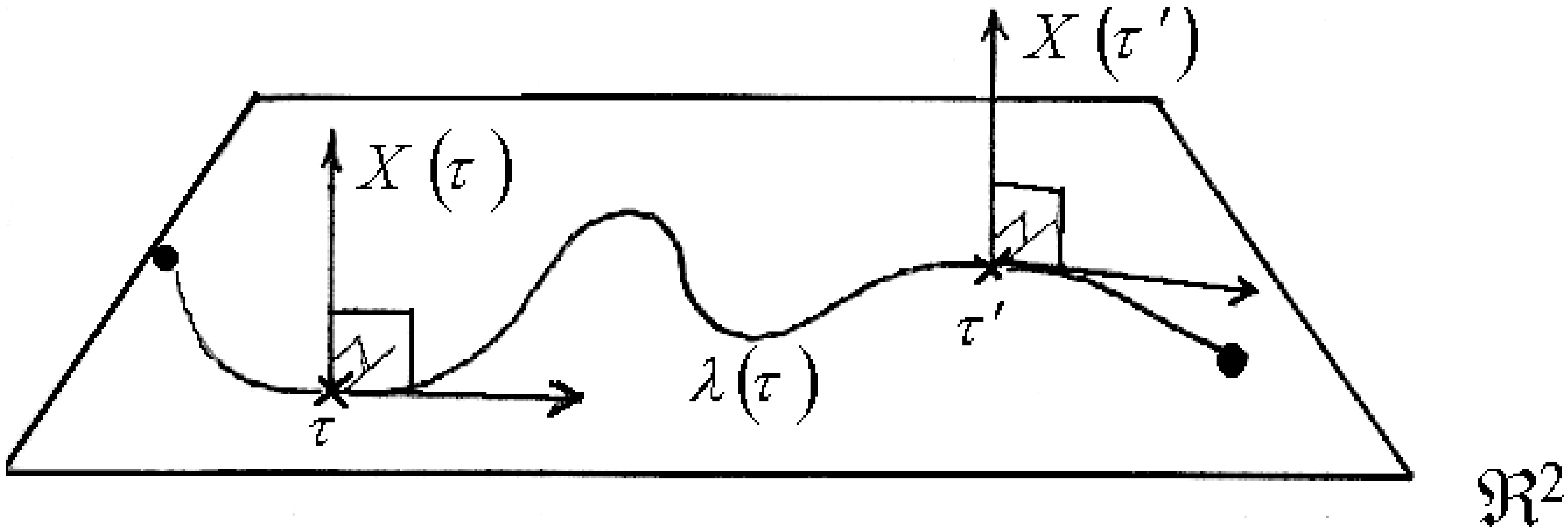
To fix this problem, we need a procedure for transporting the components of A from P to Q.



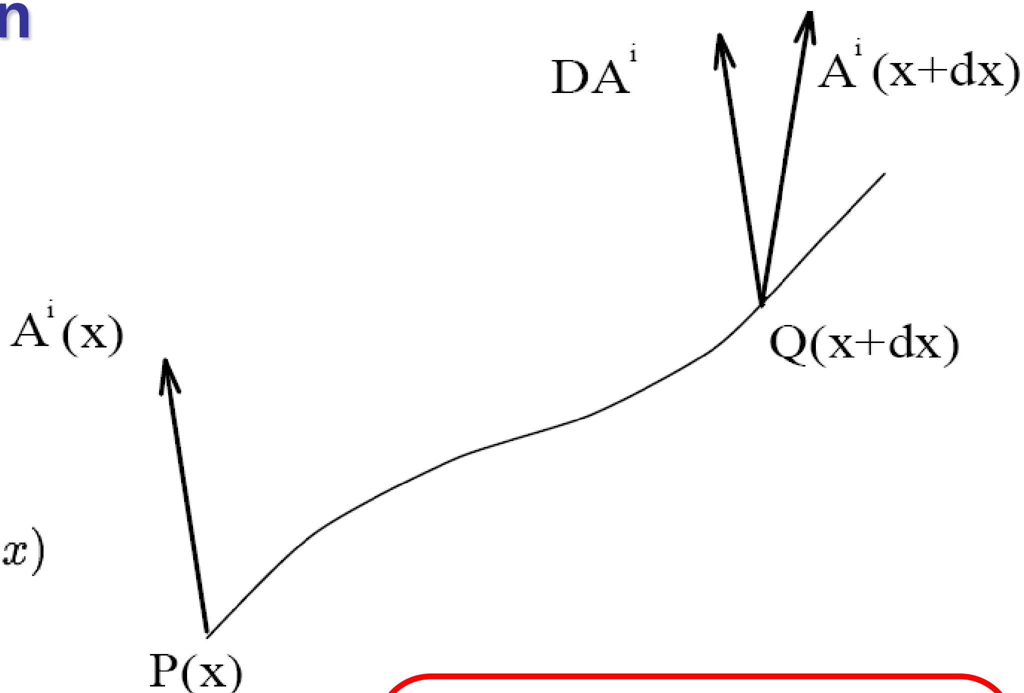
Covariant differentiation

We call this procedure **Parallel Transport**

A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve



Covariant differentiation



We can write

$$DA^i(x+dx) = A^i(x) + \delta A^i(x)$$

where

$$\delta A^i(x) = -\Gamma^i_{jk} A^j dx^k$$

$$\frac{\partial \vec{e}_i}{\partial x^k} = \Gamma^j_{ik} \, \vec{e}_j$$

Christoffel symbols, connecting the basis vectors at Q to those at P

Covariant differentiation

We can now define the **covariant derivative** of a vector (which is a tensor)

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{jk} A^j$$

Ordinary partial derivative

We want to define physical laws in terms of covariant derivatives so that they are valid in any coordinate system.

Geodesics

We can now provide a more mathematical basis for the phrase "spacetime tells matter how to move".

One can define a geodesic as a **curve along which the tangent vector to the curve is parallel-transported.** In other words, if one parallel transports a tangent vector along a geodesic, it remains a tangent vector.

The covariant derivative of a tangent vector, along the geodesic is identically zero, i.e.

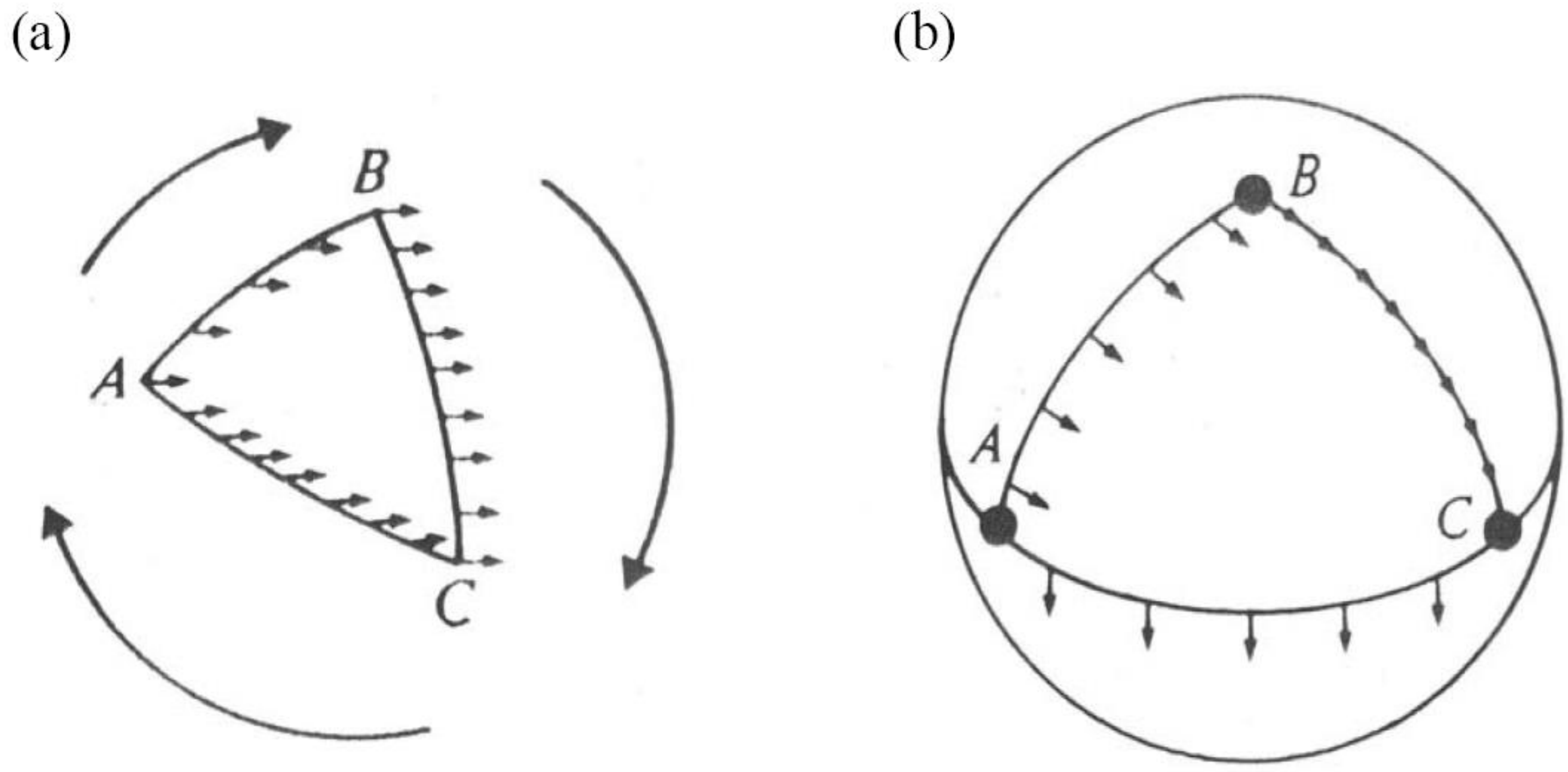
$$\vec{\nabla}_{\vec{U}}\vec{U} = 0$$

4. Spacetime curvature in GR (pgs.33 – 37)

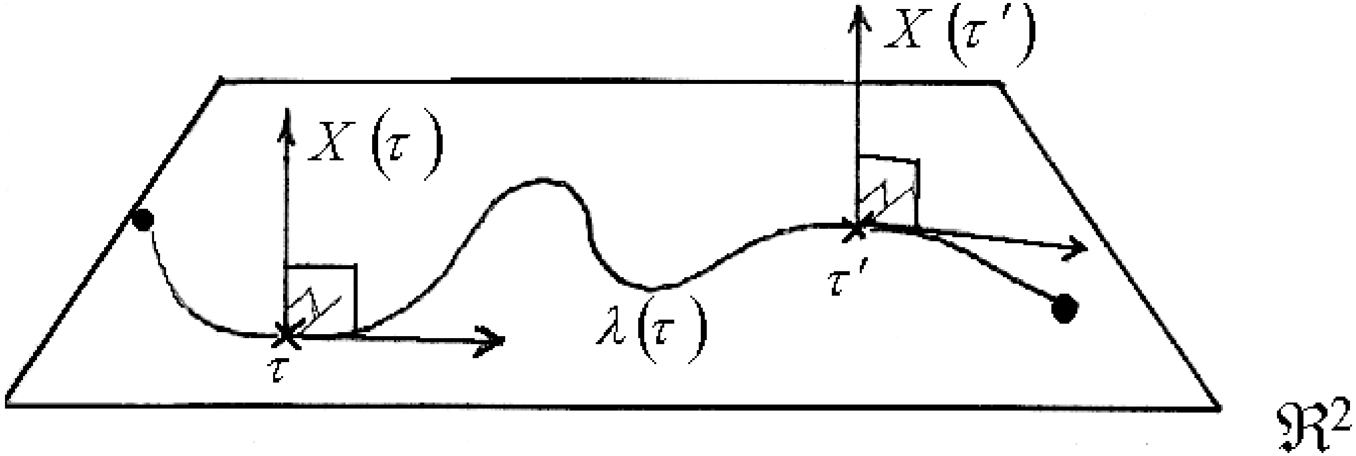
This is described by the **Riemann-Christoffel tensor**, which depends on the metric and its first and second derivatives.

We can derive the form of the R-C tensor in several ways

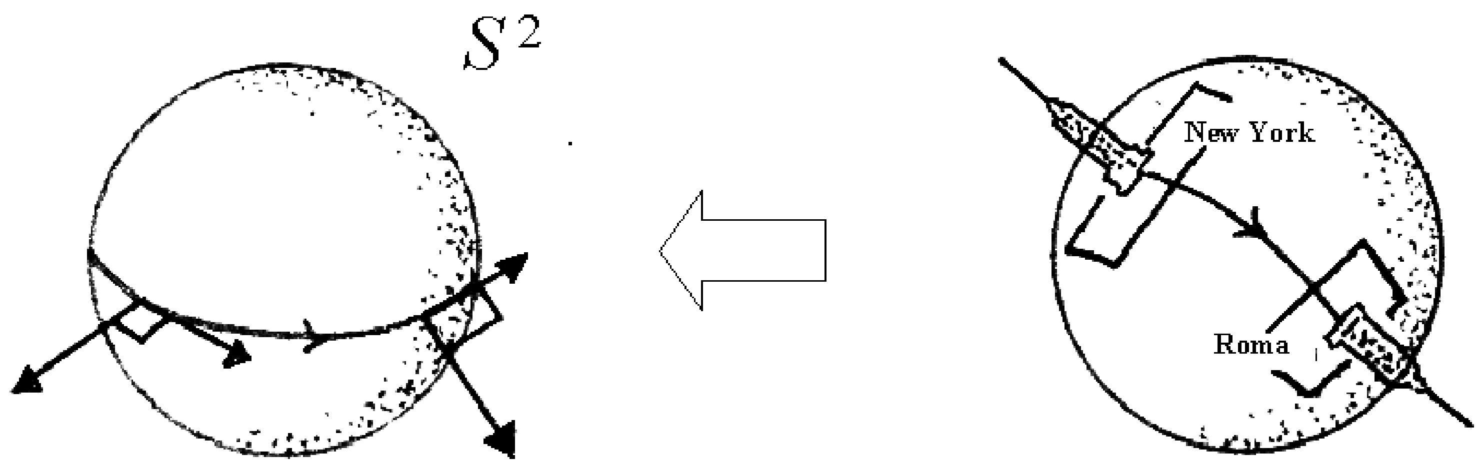
1. by parallel transporting of a vector around a closed loop in our manifold
2. by considering the commutator of the second order covariant derivative of a vector field
3. by computing the deviation of two neighbouring geodesics in our manifold

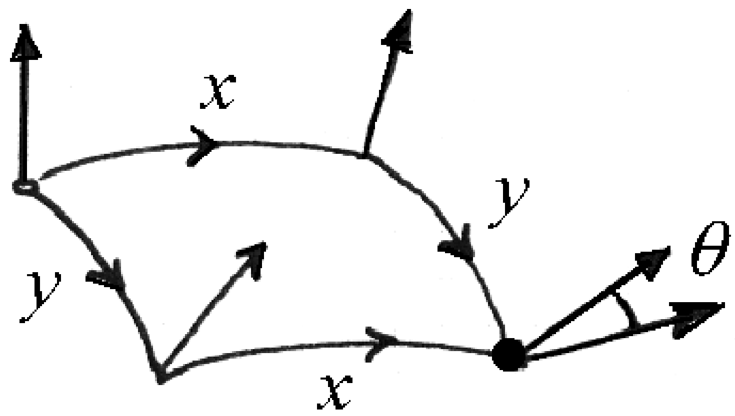


(a)
A
B
C
(b)
A
B
C



In a flat manifold, parallel transport does not rotate vectors, while on a curved manifold it *does*.





After parallel transport around a closed loop on a curved manifold, the vector does not come back to its original orientation but it is rotated through some angle.

The R-C tensor is related to this angle.

$$R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\gamma} + \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma}$$

If spacetime is flat then, for all indices $R^{\mu}{}_{\alpha\beta\gamma} = 0$

5. Einstein's Equations (pgs.38 – 45)

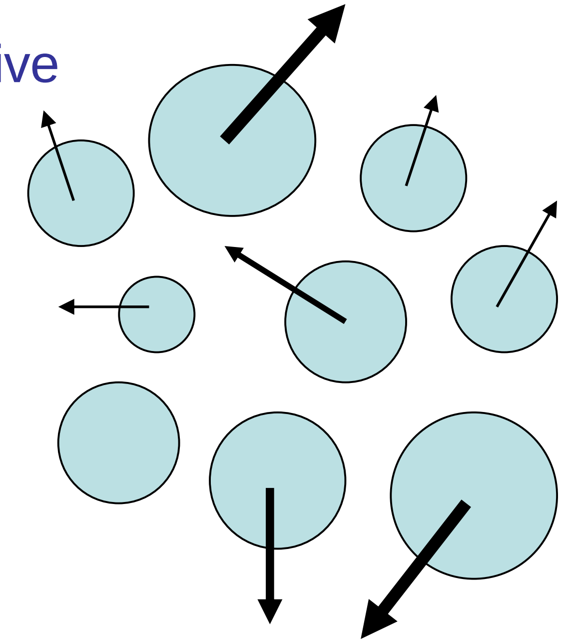
What about "matter tells spacetime how to curve"?...

The source of spacetime curvature is the **Energy-momentum tensor** which describes the presence and motion of gravitating matter (and energy).

We consider the E-M tensor for a **perfect fluid**

In a fluid description we treat our physical system as a smooth continuum, and describe its behaviour in terms of locally averaged properties in each ***fluid element****.*

Each fluid element may possess a **bulk motion** with respect to the rest of the fluid, and this relative motion may be non-uniform.



Particles within the fluid element will not be at rest:

1. Pressure (c.f. molecules in an ideal gas)
2. Heat conduction (energy exchange with neighbours)
3. Viscous forces (shearing of fluid)

Perfect Fluid if each fluid element has no heat conduction or viscous forces, only **pressure**.

Energy momentum tensor for a perfect fluid

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Pressure due to random motion of particles in fluid element

Conservation of energy and momentum requires that

$$T^{\alpha\beta}_{;\beta} = 0$$

So how does "matter tell spacetime how to curve"?...

Einstein's Equations

BUT the E-M tensor is of rank 2, whereas the R-C tensor is of rank 4.

Einstein's equations involve contractions of the R-C tensor.

Define the **Ricci tensor** by

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma}$$

and the **curvature scalar** by

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

We can raise indices via $$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta}$$

and define the Einstein tensor $$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

We can show that $$G^{\mu\nu}_{\;;\nu} = 0$$

so that $$T^{\mu\nu}_{\;;\nu} = G^{\mu\nu}_{\;;\nu}$$

Einstein took as solution the form

$$G^{\mu\nu} = kT^{\mu\nu}$$

where we can determine the constant k by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion. In fact k turns out to equal $8\pi G/c^4$.

Solving Einstein's equations

Given the metric, we can compute the Christoffel symbols, then the geodesics of 'test' particles.

We can also compute the R-C tensor, Einstein tensor and E-M tensor.

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What about the other way around?...

Highly non-trivial problem, in general intractable, but given E-M tensor can solve for metric in some special cases.

e.g. Schwarzschild solution, for the spherically symmetric static spacetime exterior to a mass M

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Coordinate singularity at $r=2M$

Geodesics for the Schwarzschild metric

Radial geodesic

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right)$$

Changing the dependent variable from r to u and the independent variable from τ to ϕ, our radial geodesic equation reduces to

$$h^2\left(\frac{du}{d\phi}\right)^2 = (k^2 - 1) - h^2u^2 + 2Mu\left(1 + h^2u^2\right)$$

or

$$\frac{d^2u}{d\phi^2} = -u + \frac{M}{h^2} + 3Mu^2$$

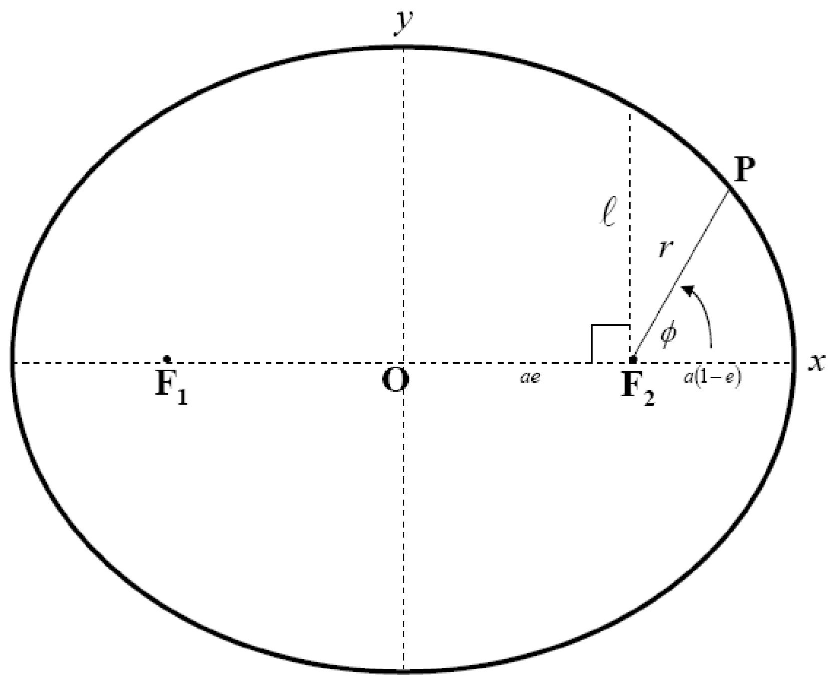
Extra term, only in GR

e.g. for the Earth's orbit the ratio

$$\frac{3Mu^2}{M/h^2} \simeq 3 \times 10^{-8}$$

Newtonian solution:

Elliptical orbit



Foci at $\mathbf{F}_1$ and $\mathbf{F}_2$

$\mathbf{O}x = a =$ semi-major axis

$\mathbf{O}y = b =$ semi-minor axis

$$b^2 = a^2(1 - e^2)$$

$e =$ eccentricity

P defined by

$$r = \frac{\ell}{1 + e\cos\phi}$$

$\ell =$ semi-latus rectum

$= b^2 / a$

GR solution:

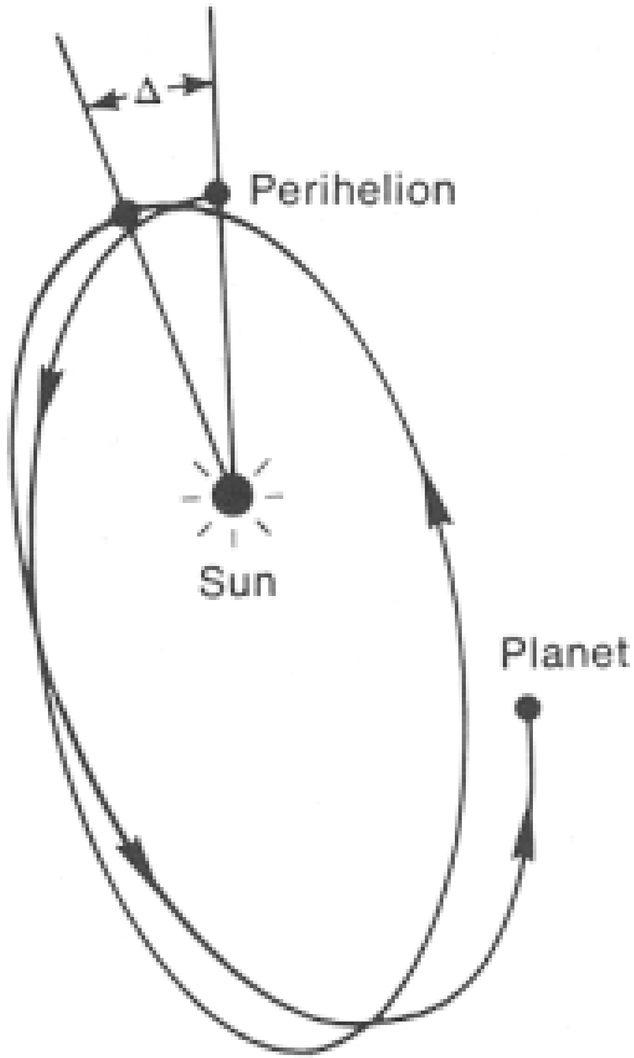
Precessing ellipse

$$u = \frac{M}{h^2}\left[1 + e\cos\left(1 - \frac{3M^2}{h^2}\right)\phi\right]$$

Here

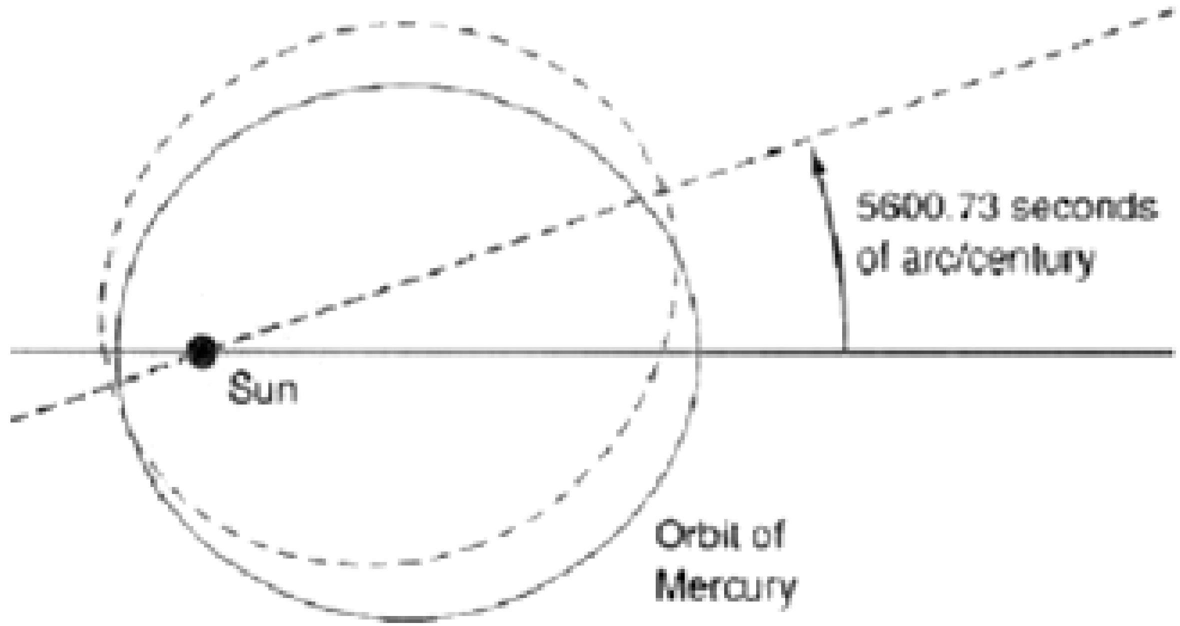
$$P = \frac{2\pi}{1 - 3M^2/h^2} > 2\pi$$

$$\Delta = \frac{6\pi M}{a(1-e^2)}$$



GR solution:

Precessing ellipse



$$\Delta = \frac{6\pi M}{a(1-e^2)}$$

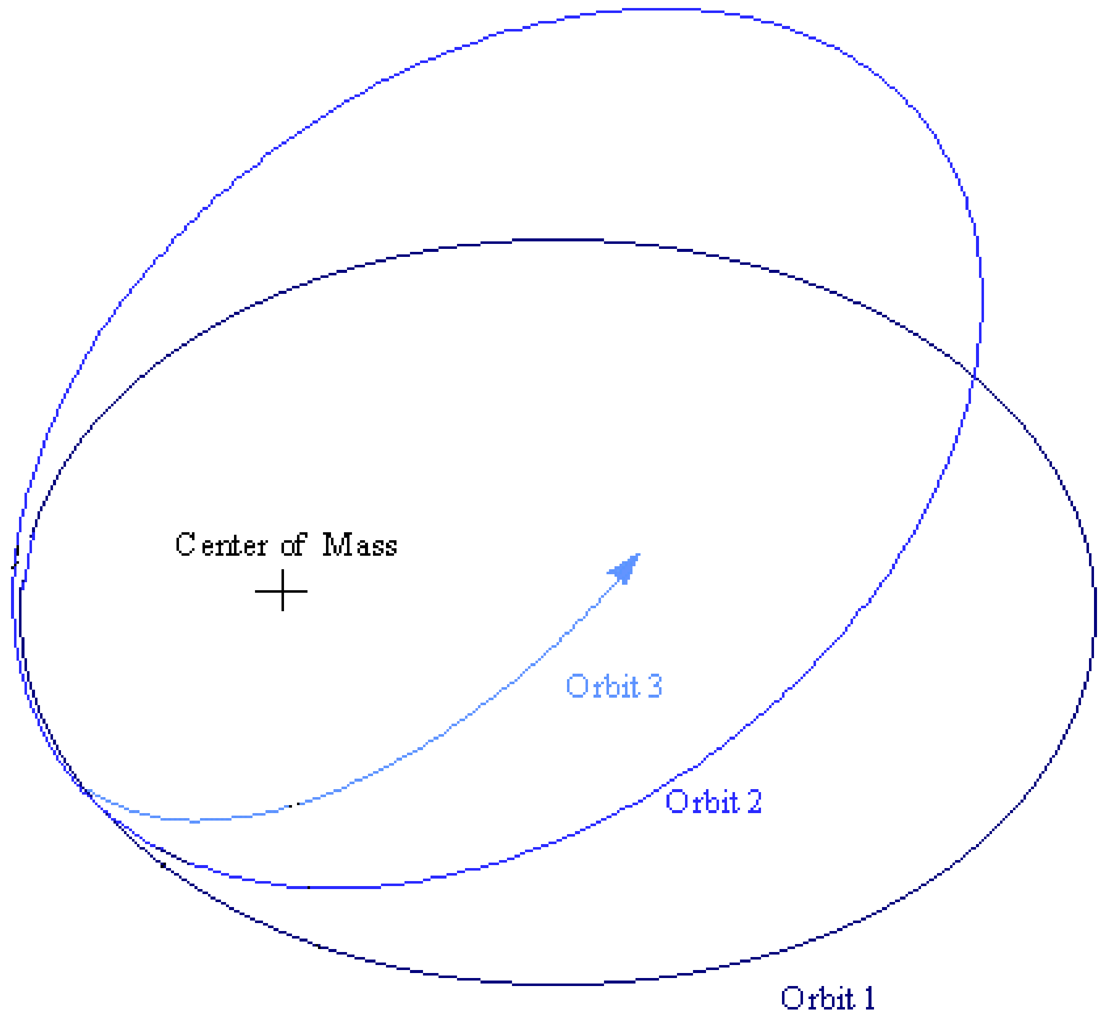
If we apply this equation to the orbit of Mercury, we obtain a perihelion advance which builds up to about 43 seconds of arc per century.

GR solution:

Precessing ellipse

Seen much more dramatically in the **binary pulsar** PSR 1913+16.

Periastron is advancing at a rate of ~4 degrees per year!



Gravitational light deflection in GR

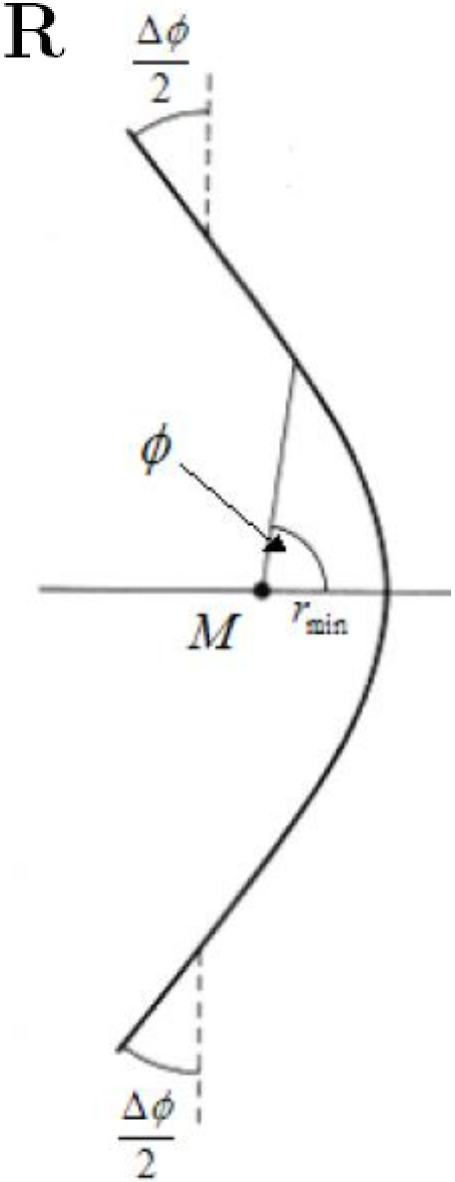
Radial geodesic for a photon

$$\left(\frac{dr}{d\lambda}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{2Mh^2}{r^3}$$

or
$$\frac{d^2u}{d\phi^2} + u = 3Mu^2$$

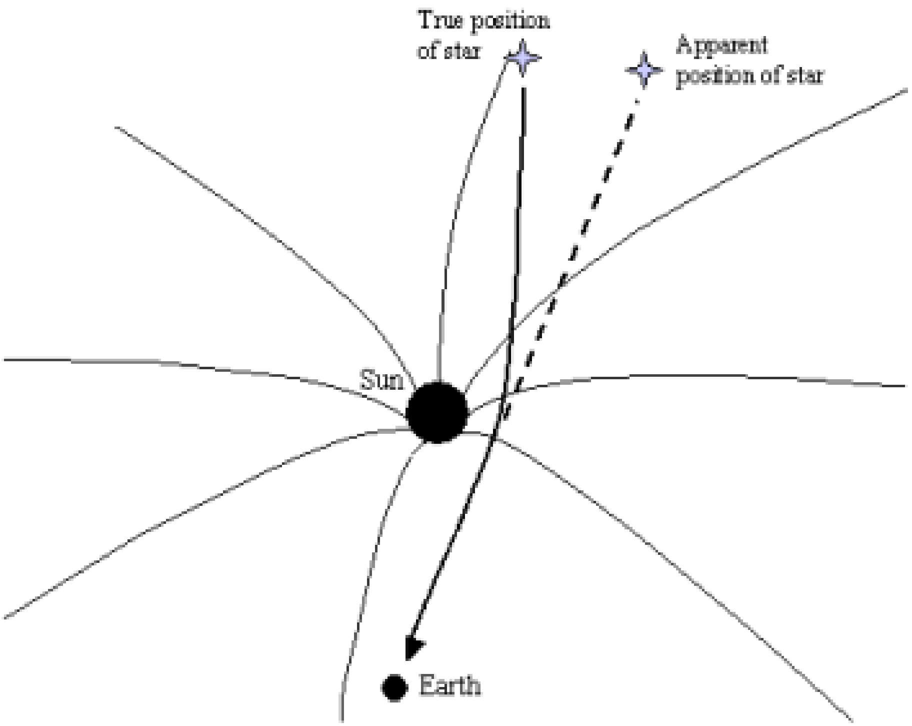
Solution reduces to
$$u = -\frac{\Delta\phi}{2r_{\rm min}} + \frac{2M}{r_{\rm min}^2}$$

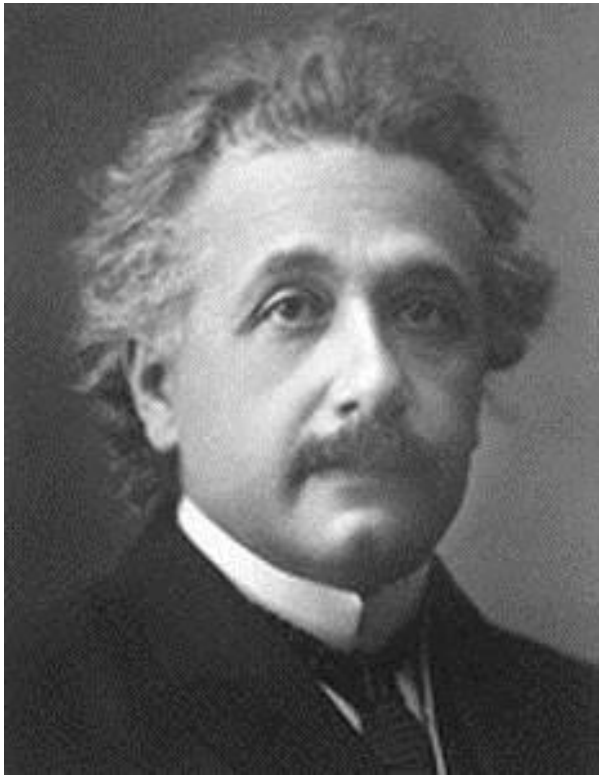
So that asymptotically
$$\Delta\phi = \frac{4M}{r_{\rm min}} \equiv \frac{4GM}{c^2 r_{\rm min}}$$



This is exactly twice the deflection angle predicted by a Newtonian treatment. If we take $r_{\min}$ to be the radius of the Sun (which would correspond to a light ray grazing the limb of the Sun from a background star observed during a total solar eclipse) then we find that

$$\Delta\phi = \frac{4 \times 1.5 \times 10^3}{6.95 \times 10^8} = 8.62 \times 10^{-6} \text{ radians} \quad = \quad 1.77 \text{ arcsec}$$



1919 expedition, led by Arthur Eddington, to observe total solar eclipse, and measure light deflection.

GR passed the test!

6. Wave Equation for Gravitational Radiation (pgs.46 – 57)

Weak gravitational fields

In the absence of a gravitational field, spacetime is flat. We define a weak gravitational field as one is which spacetime is 'nearly flat'

i.e. we can find a coord system such that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where

$$\eta_{\alpha\beta} = \mathrm{diag}\ (-1, 1, 1, 1)$$

$$|h_{\alpha\beta}| << 1 \ \text{ for all } \ \alpha \ \text{ and } \ \beta$$

This is known as a Nearly Lorentz coordinate system.

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

1) Background Lorentz transformations

Hence, our original nearly Lorentz coordinate system remains nearly Lorentz in the new coordinate system. In other words, a spacetime which looks nearly flat to one observer still looks nearly flat to any other observer in uniform relative motion with respect to the first observer.

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

2) Gauge transformations

The above results tell us that – once we have identified a coordinate system which is nearly Lorentz – we can add an arbitrary small vector ξ^α to the coordinates x^α without altering the validity of our assumption that spacetime is nearly flat. We can, therefore, choose the components ξ^α to make Einstein's equations as simple as possible. We call this step choosing a **gauge** for the problem – a name which has resonance with a similar procedure in electromagnetism – and coordinate transformations of this type given by equation are known as **gauge transformation**. We will consider below specific choices of gauge which are particularly useful.

Einstein's equations for a weak gravitational field

To first order, the R-C tensor for a weak field reduces to

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right)$$

and is invariant under gauge transformations.

Similarly, the Ricci tensor is $$R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu\ ,\nu\alpha} + h^{\alpha}_{\nu\ ,\mu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right)$$

where $$h \equiv h^{\alpha}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}$$

$$h_{\mu\nu,\alpha}{}^{,\alpha} \;=\; \eta^{\alpha\sigma}\left(h_{\mu\nu,\alpha}\right)_{,\sigma} \;=\; \eta^{\alpha\sigma} h_{\mu\nu,\alpha\sigma}$$

The Einstein tensor is the (rather messy) expression

$$G_{\mu\nu} = \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right]$$

but we can simplify this by introducing $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$

So that

$$G_{\mu\nu} = -\frac{1}{2}\left[\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} + \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} - \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} - \overline{h}_{\nu\alpha,\mu}{}^{,\alpha}\right]$$

And we can choose the **Lorenz gauge** to eliminate the last 3 terms

In the Lorenz gauge, then Einstein's equations are simply

$$-\bar{h}_{\mu\nu,\alpha}{}^{,\alpha} = 16\pi T_{\mu\nu}$$

And in free space this gives

$$\bar{h}_{\mu\nu,\alpha}{}^{,\alpha} = 0$$

Writing $\bar{h}_{\mu\nu,\alpha}{}^{,\alpha} \equiv \eta^{\alpha\alpha}\bar{h}_{\mu\nu,\alpha\alpha}$

or

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0$$

In the **Lorenz gauge**, the Einstein's equations simplify in free space to

$$\left(-\frac{\partial^2}{\partial t^2} + c^2 \nabla^2 \right) \bar{h}_{\mu\nu} = 0$$

Modified form of metric perturbation

This is a key result. It has the mathematical form of a wave equation, propagating with speed *c*.
We have shown that the metric perturbations – the 'ripples' in spacetime produced by disturbing the metric – propagate at the speed of light as waves in free space.

7. The Transverse Traceless Gauge (pgs.57 – 62)

Simplest solutions of our wave equation are **plane waves**

$$\overline{h}_{\mu\nu} = \mathrm{Re}\left[A_{\mu\nu}\,\exp\left(ik_\alpha x^\alpha\right)\right]$$

Wave amplitude ($A_{\mu\nu}$)

Wave vector (k_α)

Note the wave amplitude is symmetric → 10 independent components.

Also, easy to show that

$$k_\alpha\, k^\alpha = 0$$

i.e. the wave vector is a **null** vector

Suppose we orient our coordinate axes so that the plane wave is travelling in the positive z direction. Then

$$k^t = \omega\,, \quad k^x = k^y = 0\,, \quad k^z = \omega$$

and

$$A_{\alpha z} = 0 \quad \text{for all } \alpha$$

i.e. there is no component of the metric perturbation in the direction of propagation of the wave. This explains the origin of the 'Transverse' part

So in the transverse traceless gauge,

$$\bar{h}_{\mu\nu}^{(\mathrm{TT})} = A_{\mu\nu}^{(\mathrm{TT})} \cos\left[\omega(t - z)\right]$$

where

$$A_{\mu\nu}^{(\mathrm{TT})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx}^{(\mathrm{TT})} & A_{xy}^{(\mathrm{TT})} & 0 \\ 0 & A_{xy}^{(\mathrm{TT})} & -A_{xx}^{(\mathrm{TT})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, since the perturbation is traceless $\bar{h}_{\alpha\beta}^{(\mathrm{TT})} = h_{\alpha\beta}^{(\mathrm{TT})}$

8. Effect of Gravitational Waves on Free Particles (pgs.63 – 75)

Choose Background frame in which test particle initially at rest. Set up coordinate system according to the TT gauge.

Coordinates do not change, but adjust themselves as wave passes so that particles remain 'attached' to initial positions.

Coordinates are frame-dependent labels.

What about **proper distance** between neighbouring particles?

Consider two test particles, both initially at rest, one at origin and the other at $x = \epsilon\,,\; y = z = 0$

$$\Delta\ell = \int \left| g_{\alpha\beta} dx^\alpha dx^\beta \right|^{1/2}$$

i.e. $$\Delta\ell = \int_0^\epsilon \left| g_{xx} \right|^{1/2} \quad \simeq \quad \sqrt{g_{xx}(x=0)}\;\epsilon$$

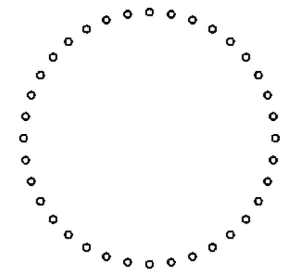
Now $$g_{xx}(x=0) = \eta_{xx} + h_{xx}^{(\mathrm{TT})}(x=0)$$

so $$\Delta\ell \simeq \left[1 + \frac{1}{2} h_{xx}^{(\mathrm{TT})}(x=0) \right] \epsilon$$

In general, this is time-varying

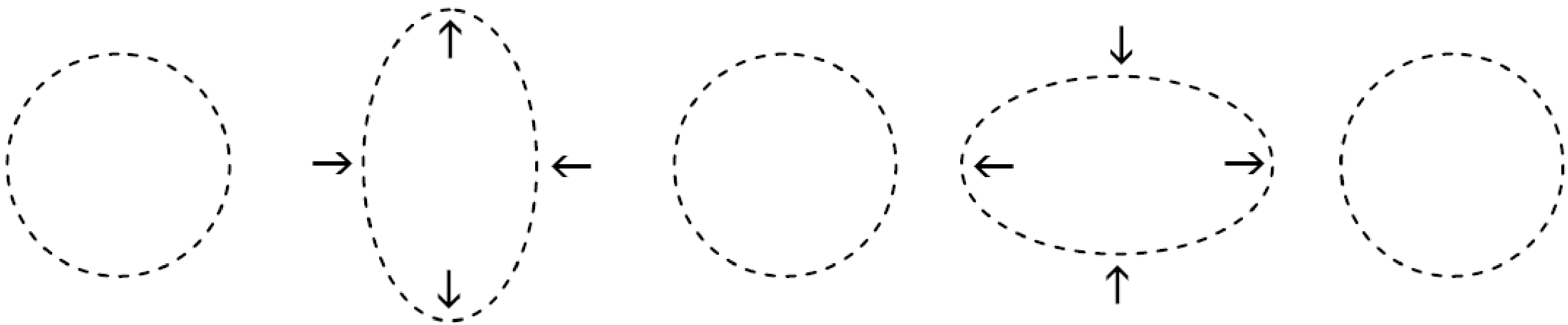
$$A_{xx}^{(\mathrm{TT})} \neq 0 \qquad A_{xy}^{(\mathrm{TT})} = 0$$

$$\xi^x = \epsilon \cos\theta \left(1 + \frac{1}{2} A_{xx}^{(\mathrm{TT})} \cos\omega t \right)$$



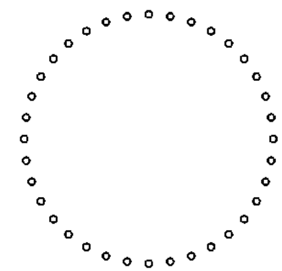
$$\xi^y = \epsilon \sin\theta \left(1 - \frac{1}{2} A_{xx}^{(\mathrm{TT})} \cos\omega t \right)$$

$A_{xx}^{(\mathrm{TT})} \neq 0$ **+ Polarisation**



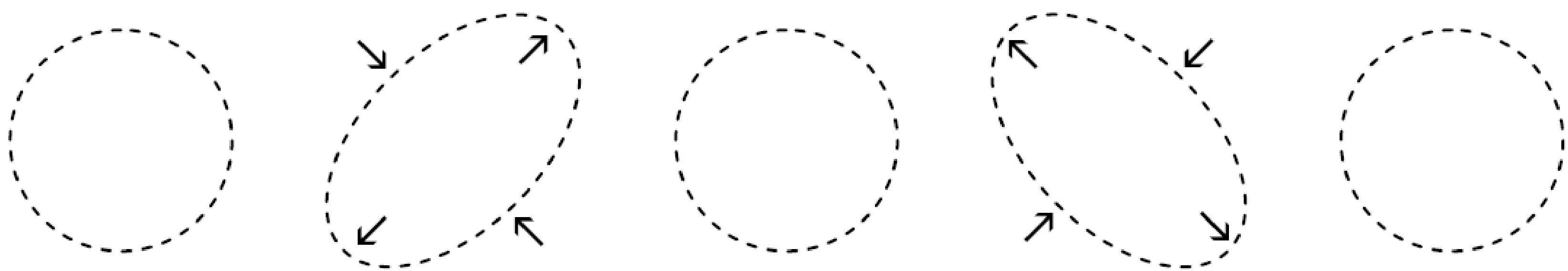
$$A_{xy}^{(\mathrm{TT})} \neq 0 \qquad A_{xx}^{(\mathrm{TT})} = 0$$

$$\xi^x = \epsilon \cos\theta + \frac{1}{2}\epsilon \sin\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t$$



$$\xi^y = \epsilon \sin\theta + \frac{1}{2}\epsilon \cos\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t$$

$A_{xy}^{(\mathrm{TT})} \neq 0$ $\quad \times$ **Polarisation**



- The two solutions, for $A_{xx}^{(TT)} \neq 0$ and $A_{xy}^{(TT)} \neq 0$ represent two independent gravitational wave **polarisation states**, and these states are usually denoted by '+' and '×' respectively. In general any gravitational wave propagating along the z-axis can be expressed as a linear combination of the '+' and '×' polarisations, i.e. we can write the wave as

$$\mathbf{h} = a\,\mathbf{e}_+ + b\,\mathbf{e}_\times$$

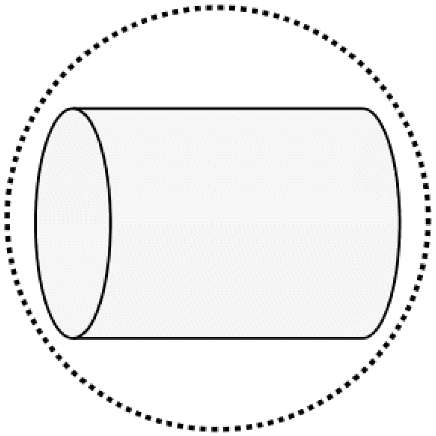
where a and b are scalar constants and the *polarisation tensors* $\mathbf{e}_+$ and $\mathbf{e}_\times$ are

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

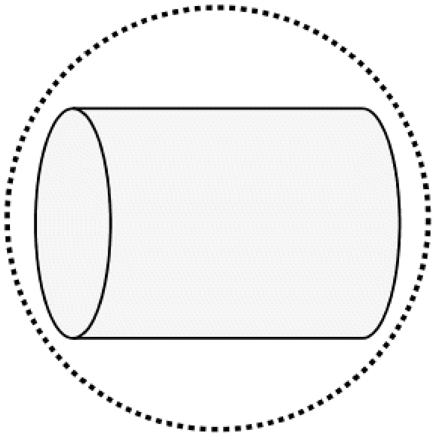
- Distortions are **quadrupolar** - consequence of fact that acceleration of geodesic deviation non-zero only for tidal gravitational field.

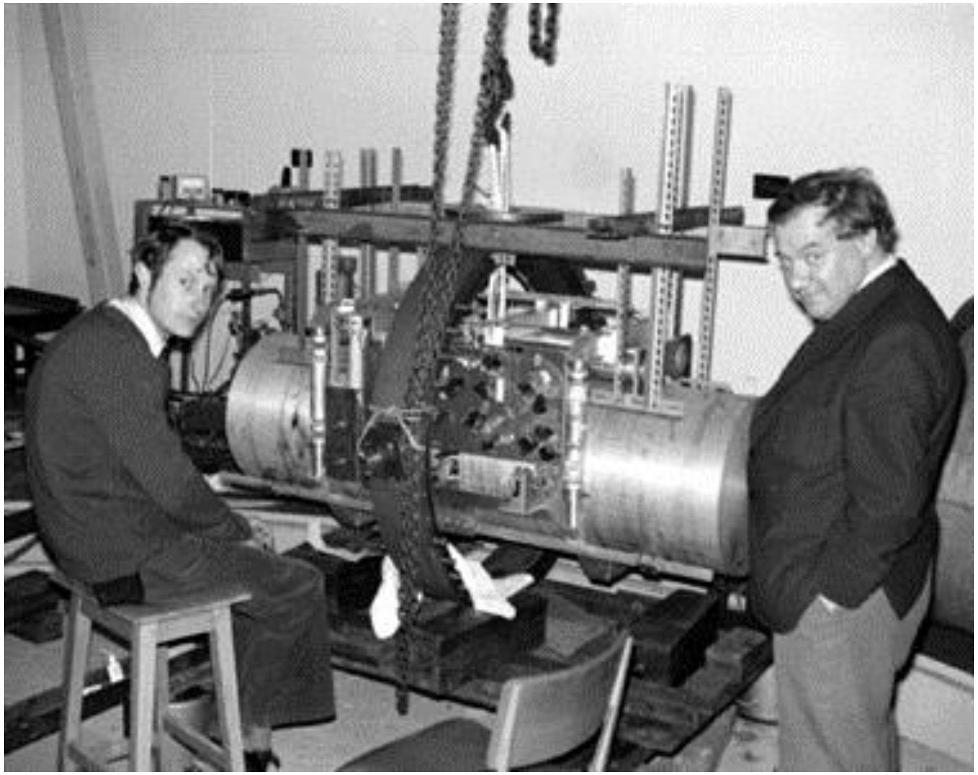
- At any instant, a gravitational wave is invariant under a rotation of 180 degrees about its direction of propagation.
  (c.f. spin states of gauge bosons; graviton must be S=2, tensor field)

Design of gravitational wave detectors

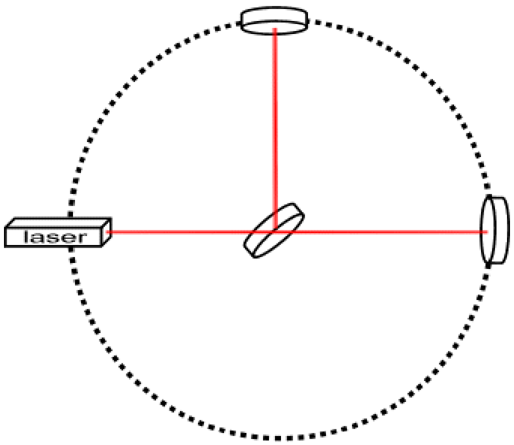


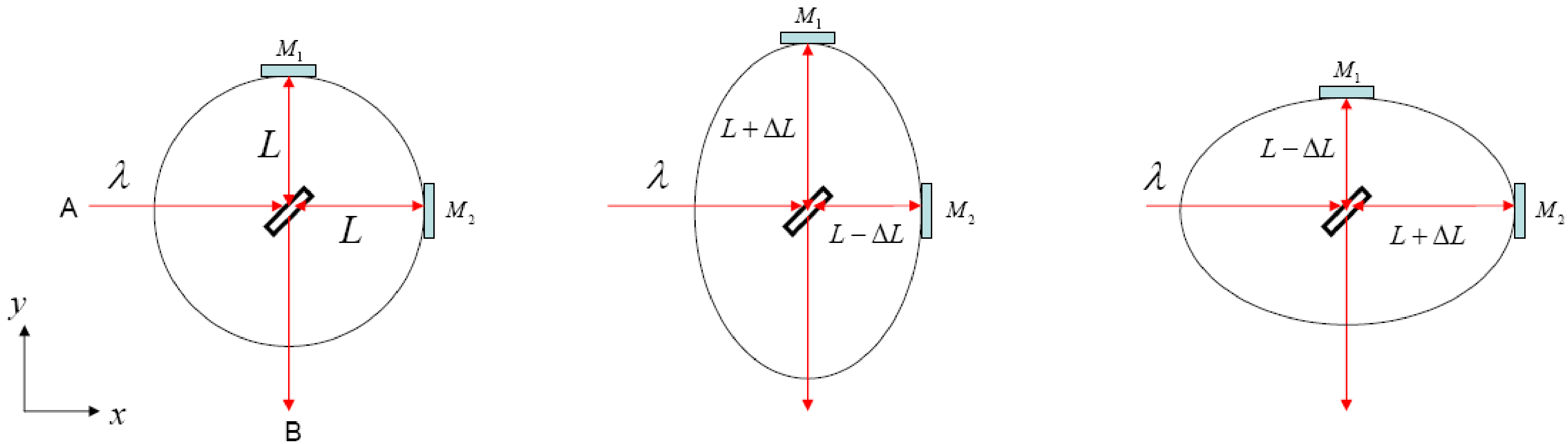
Design of gravitational wave detectors





~40 yrs on - Interferometric ground-based detectors





Gravitational wave $\mathbf{h} = h\mathbf{e}_+$ propagating along z axis.

Fractional change in proper separation

$$\frac{\Delta L}{L} = \frac{h}{2}$$

9. The Production of Gravitational Waves (pgs 76 – 80)

We can understand something important about the nature of gravitational radiation by drawing analogies with the formulae that describe electromagnetic radiation.

$$L_{\text{electric dipole}} \propto e^2\,\ddot{\mathbf{d}}^2$$

← Net electric dipole moment

Analogue is the mass dipole moment: $$\mathbf{d} = \sum_{A_i} m_i \mathbf{x}_i$$

But $$\dot{\mathbf{d}} = \sum_{A_i} m_i \dot{\mathbf{x}}_i \equiv \mathbf{p}$$

Conservation of linear momentum means no "electric dipole"

$$L_{\text{magnetic dipole}} \propto \ddot{\mu}$$

$$\mu = \sum_{q_i} (\text{position of } q_i) \times (\text{current due to } q_i)$$

Gravitational analogues?...

$$\mu = \sum_{A_i} (\mathbf{x}_i) \times (m_i \mathbf{v}_i) \equiv \mathbf{J}$$

Conservation of angular momentum implies no “magnetic dipole”

So lowest order radiation produced is “quadrupole”

Also, the quadrupole of a **spherically symmetric mass distribution** is zero.

Metric perturbations which are spherically symmetric don't produce gravitational radiation.

Example: binary neutron star system.

$$h_{\mu\nu} = \frac{2G}{c^4 r} \ddot{I}_{\mu\nu}$$

where $I_{\mu\nu}$ is the **reduced quadrupole moment** defined as

$$I_{\mu\nu} = \int \rho(\vec{r}) \left(x_\mu x_\nu - \frac{1}{3} \delta_{\mu\nu} r^2 \right) dV$$

Consider a binary neutron star system consisting of two stars both of Schwarzschild mass M, in a circular orbit of coordinate radius R and orbital frequency f.

$$I_{xx} = 2MR^2 \left[\cos^2(2\pi ft) - \frac{1}{3}\right]$$

$$I_{yy} = 2MR^2 \left[\sin^2(2\pi ft) - \frac{1}{3}\right]$$

$$I_{xy} = I_{yx} = 2MR^2 \left[\cos(2\pi ft)\sin(2\pi ft)\right]$$

Thus
$$h_{xx} = -h_{yy} = h\cos(4\pi ft)$$

$$h_{xy} = h_{yx} = -h\sin(4\pi ft)$$

where
$$h = \frac{32\pi^2 GMR^2 f^2}{c^4 r}$$

So the binary system emits gravitational waves at **twice** the Keplerian orbital frequency.

For GW150914: $M \sim 30\ M_\odot$ $r \sim 400$ Mpc

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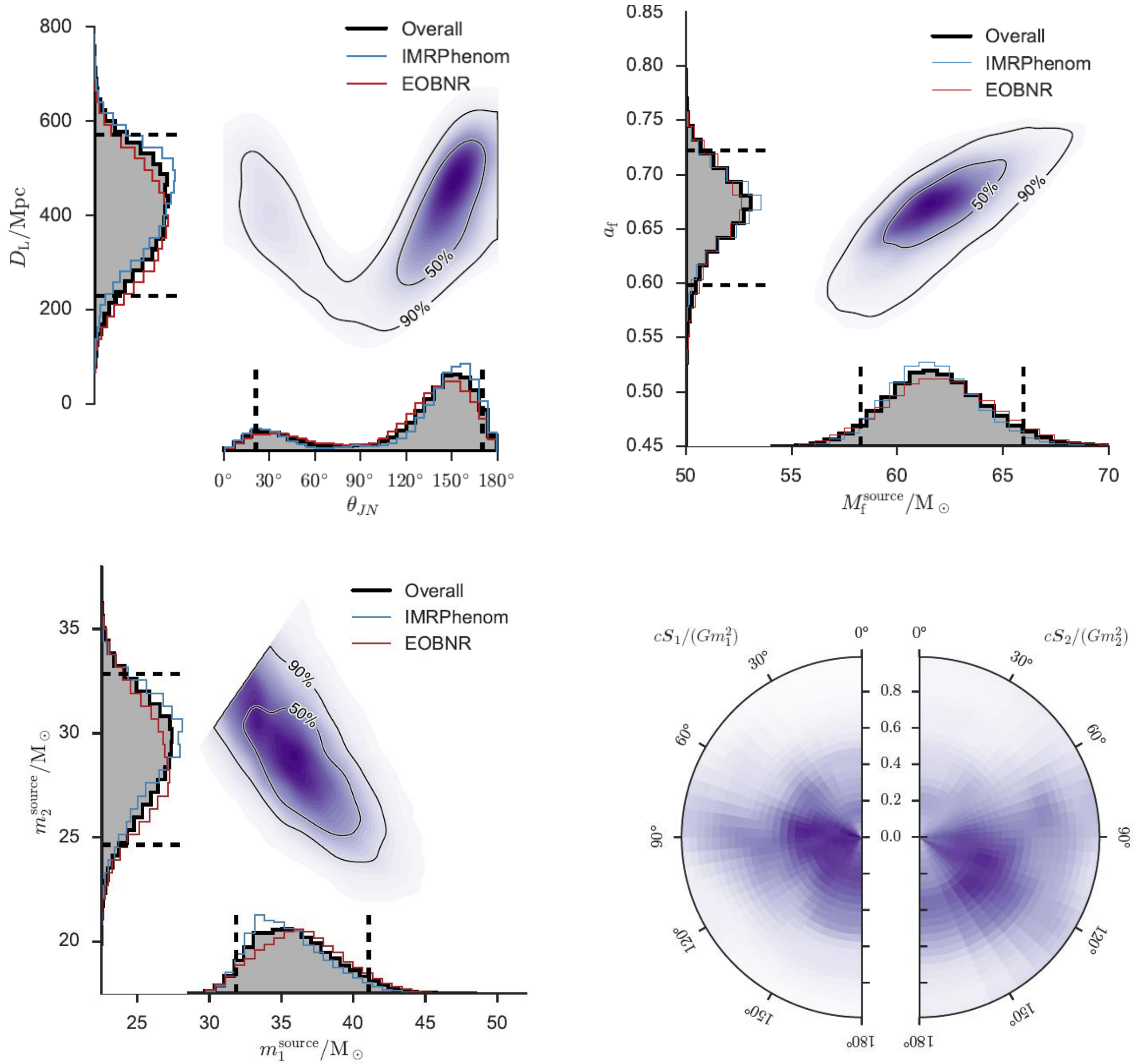
GW150914 Parameters

distance vs inclination

final mass vs final spin

component masses

component spins



arXiv:1602.03840

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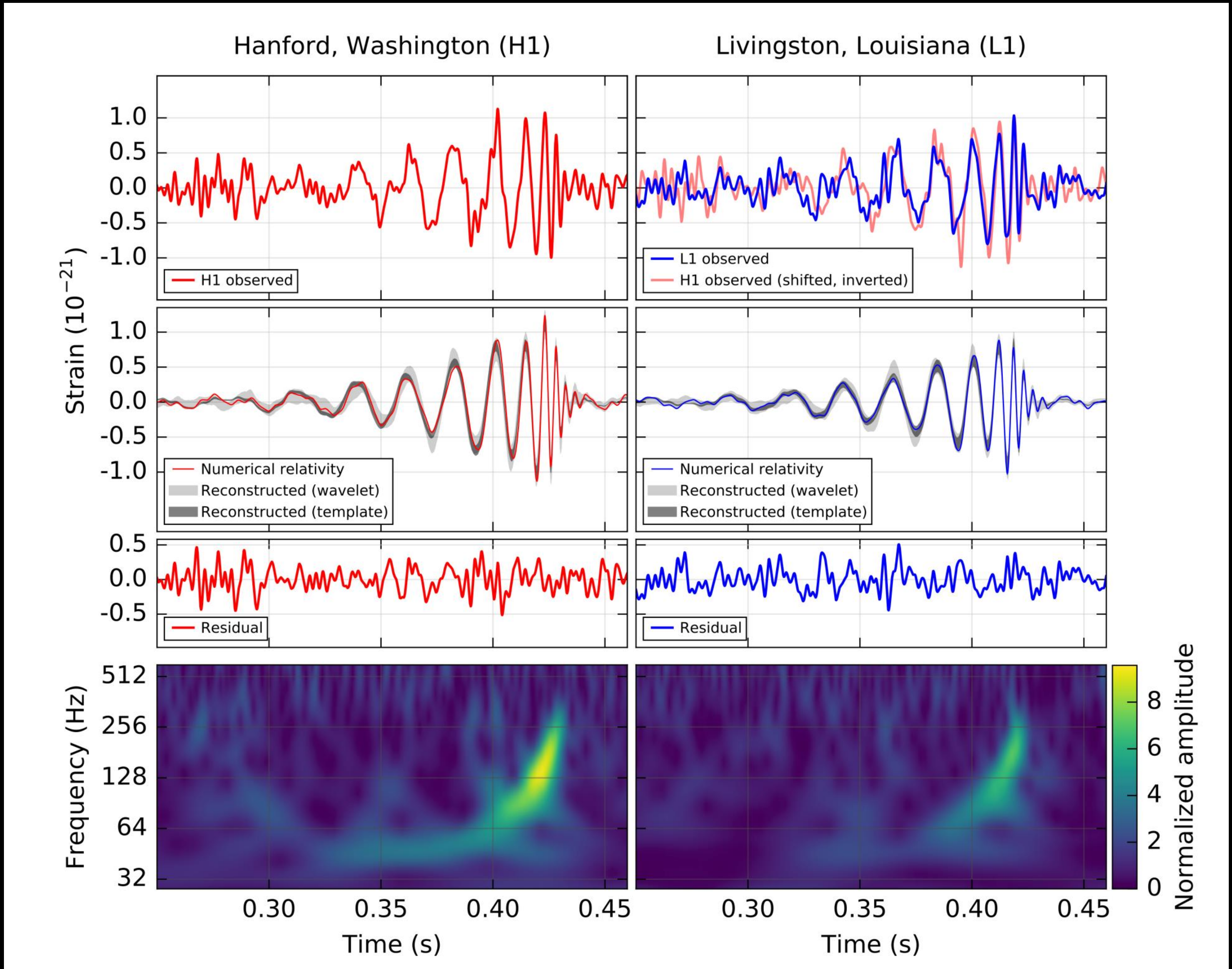
SUPA



Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
512
256
128
64
32
Normalized amplitude
Time (s)
0.30
0.35
0.40
0.45

Thus

$$h_{xx} = -h_{yy} = h\cos(4\pi f t)$$

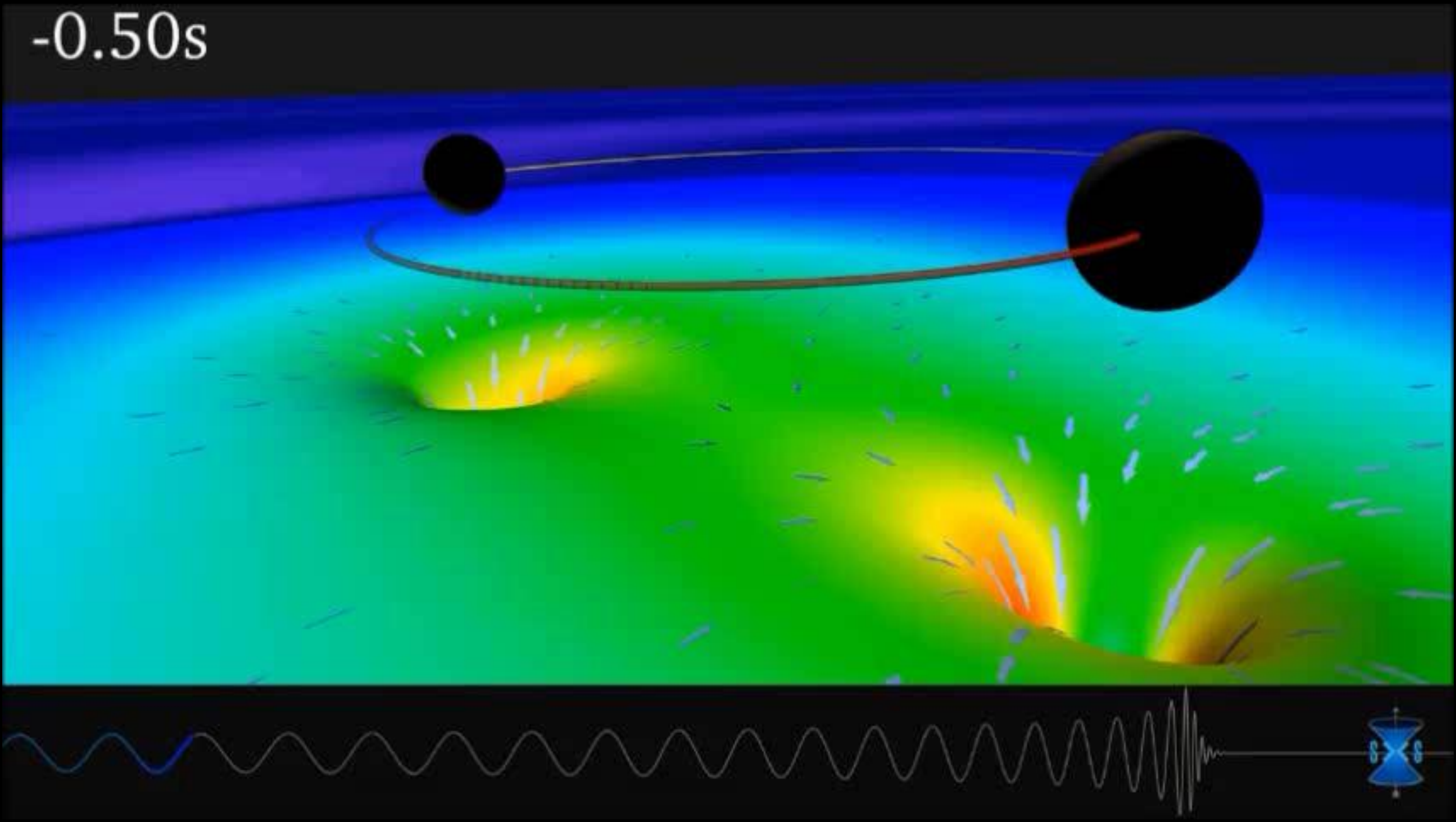
where

$$h = \frac{32\pi^2 G M R^2 f^2}{c^4 r}$$

So the binary system emits gravitational waves at **twice** the Keplerian orbital frequency.

For GW150914: $M \sim 30\ M_{\odot}$ $r \sim 400$ Mpc

$R \sim 350$ km **!!!**



https://www.ligo.caltech.edu/video/ligo20160211v10

August 5, 2016

Arxiv: 1608.01940

The basic physics of the binary black hole merger GW150914

*The LIGO Scientific Collaboration and The Virgo Collaboration**

The first direct gravitational-wave detection was made by the Advanced Laser Interferometer Gravitational Wave Observatory on September 14, 2015. The GW150914 signal was strong enough to be apparent, without using any waveform model, in the filtered detector strain data. Here those features of the signal visible in these data are used, along with only such concepts from Newtonian physics and general relativity as are accessible to anyone with a general physics background. The simple analysis presented here is consistent with the fully general-relativistic analyses published elsewhere, in showing that the signal was produced by the inspiral and subsequent merger of two black holes. The black holes were each of approximately $35\,M_{\odot}$, still orbited each other as close as ~ 350 km apart and subsequently merged to form a single black hole. Similar reasoning, directly from the data, is used to roughly estimate how far these black holes were from the Earth, and the energy that they radiated in gravitational waves.

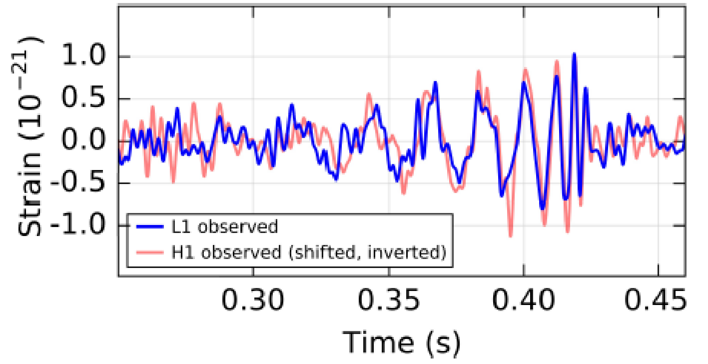


Figure 1 The instrumental strain data in the Livingston detector (blue) and Hanford detector (red), as shown in Figure 1 of [1]. Both have been bandpass- and notch-filtered. The Hanford strain has been shifted back in time by 6.9 ms and inverted. Times shown are relative to 09:50:45 Coordinated Universal Time (UTC) on September 14, 2015.

A black hole is a region of space-time where the gravitational field is so intense that neither matter nor radiation can escape. There is a natural "gravitational radius" associated with a mass m, called the Schwarzschild radius, given by

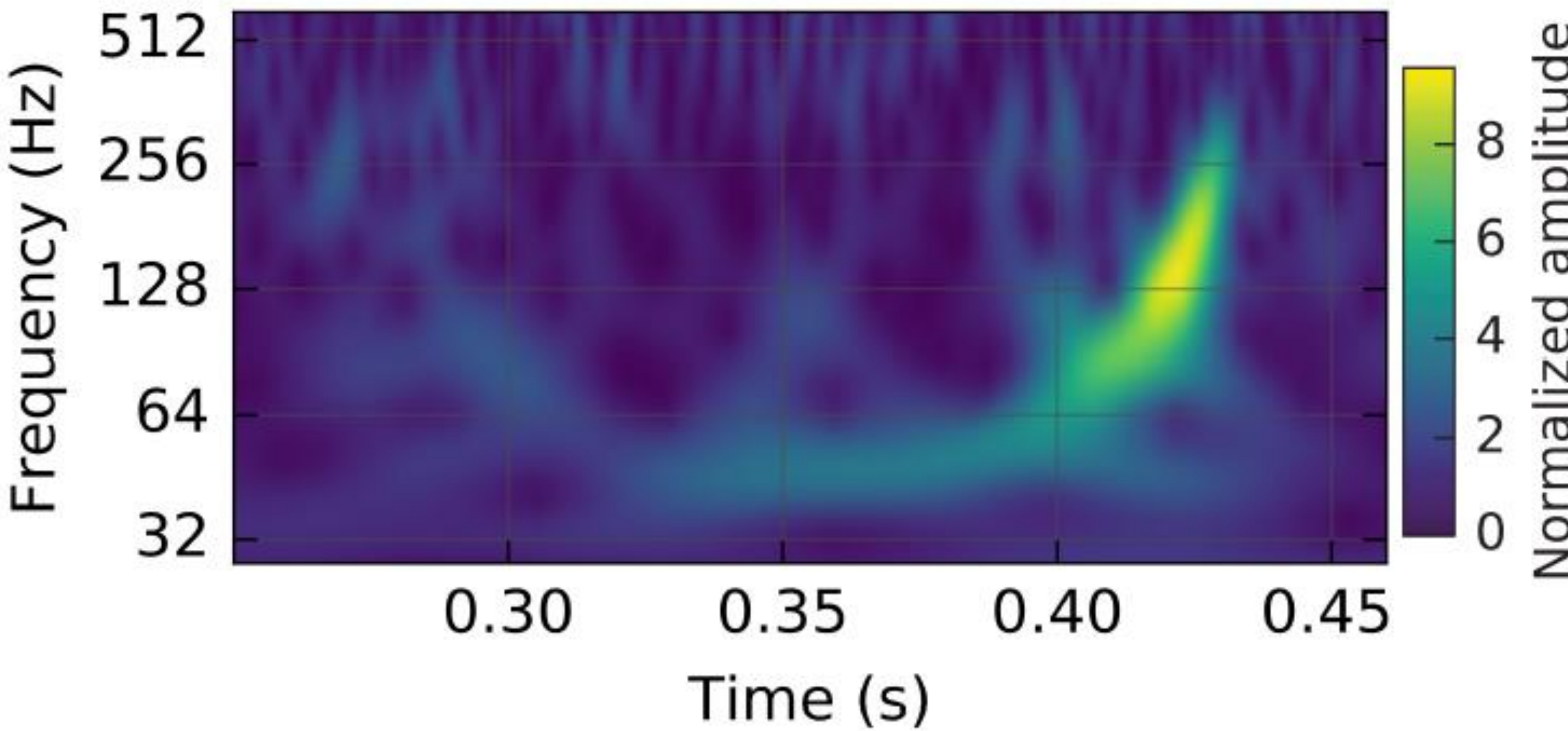
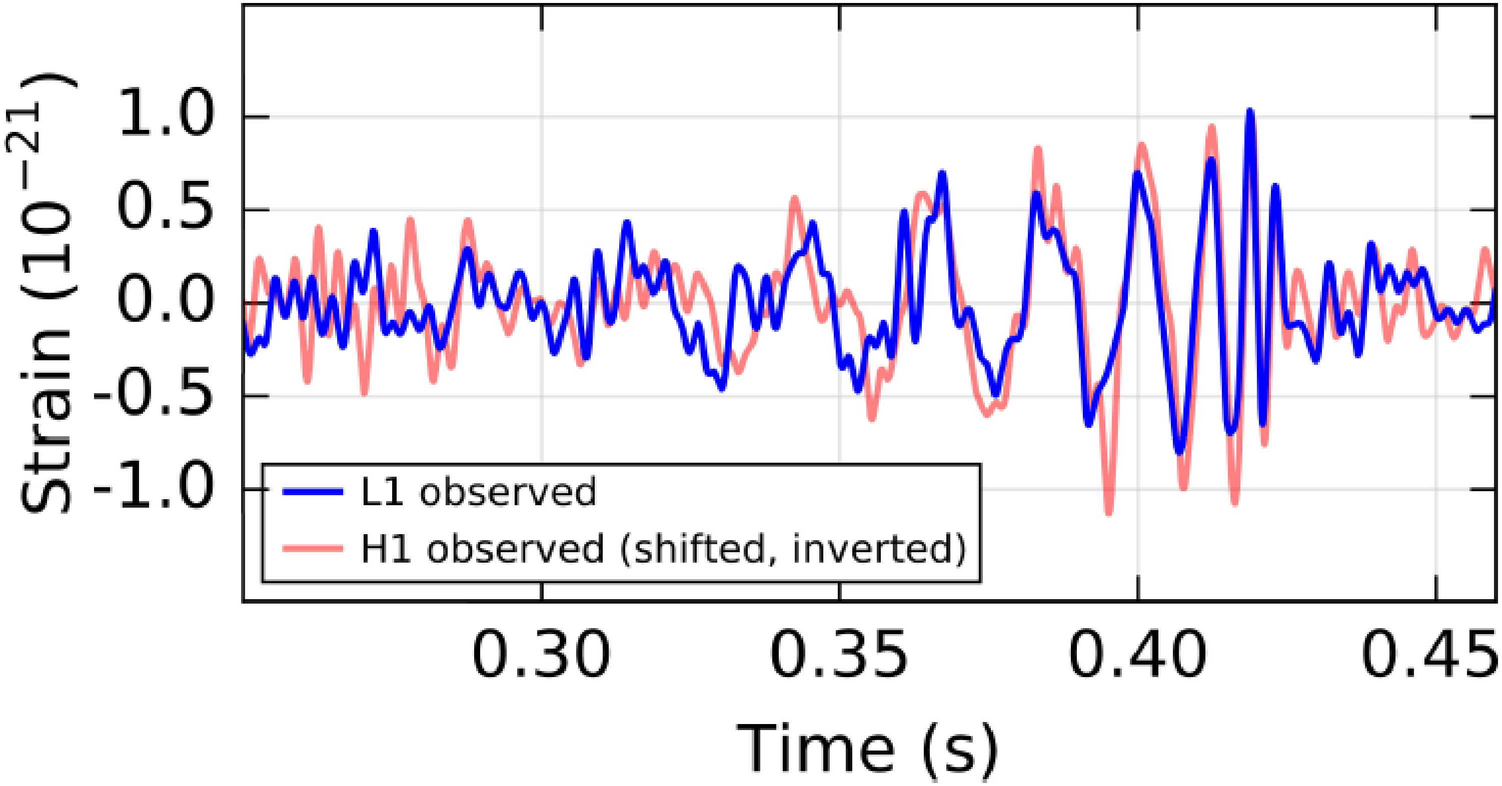


Figure 2 A representation of the strain-data as a time-frequency plot (taken from [1]), where the increase in signal frequency (“chirp") can be traced over time.



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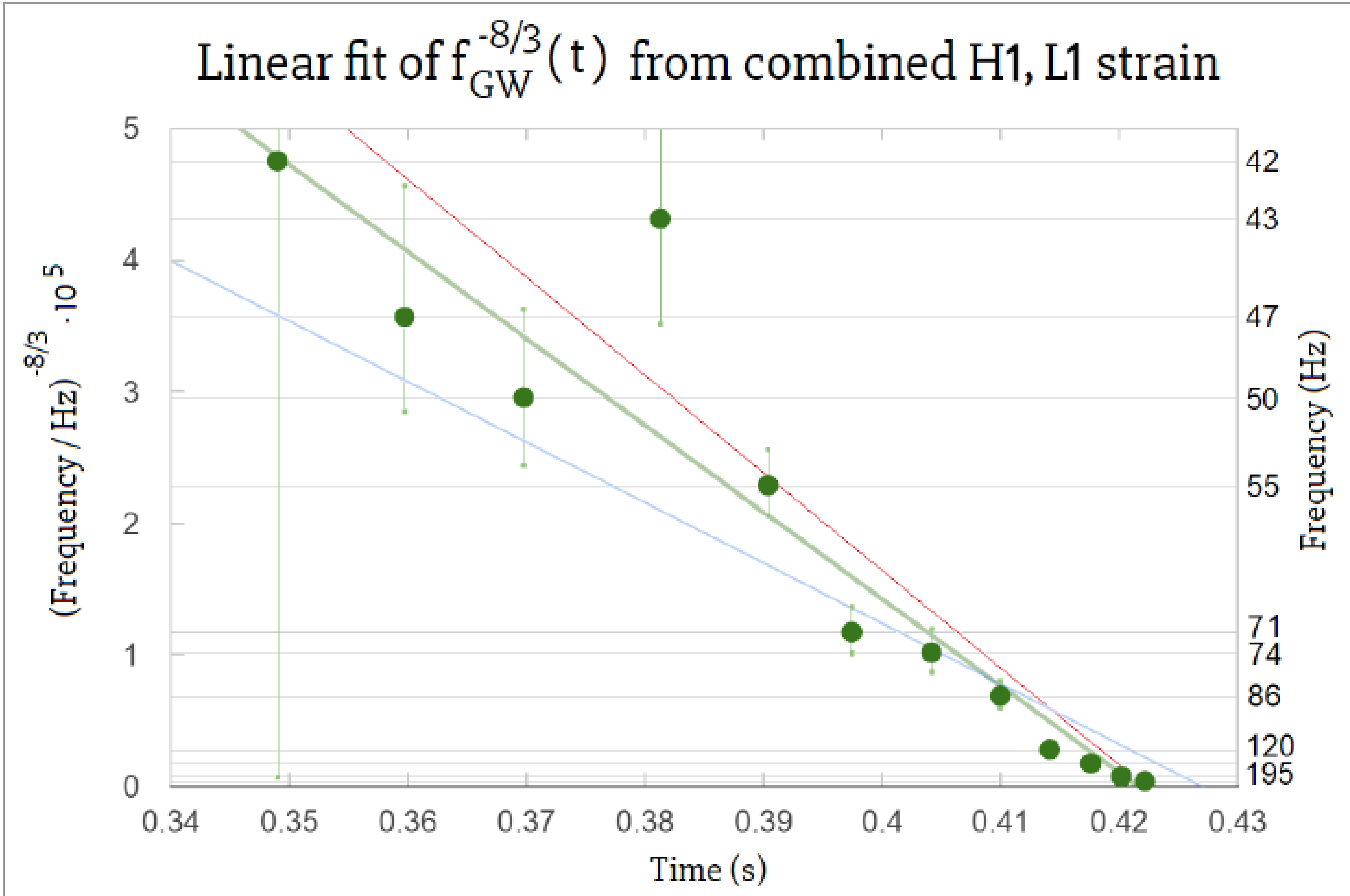
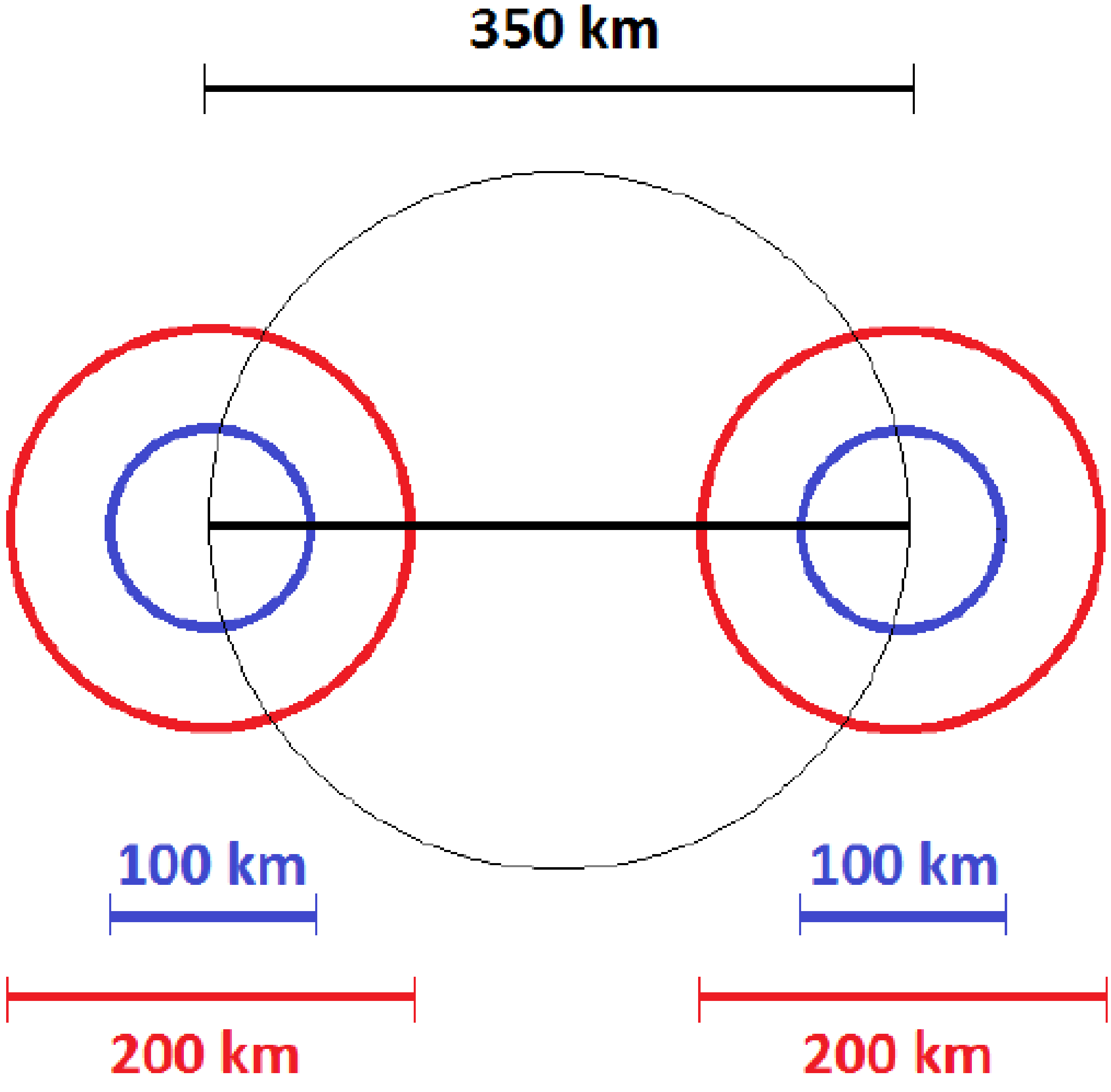
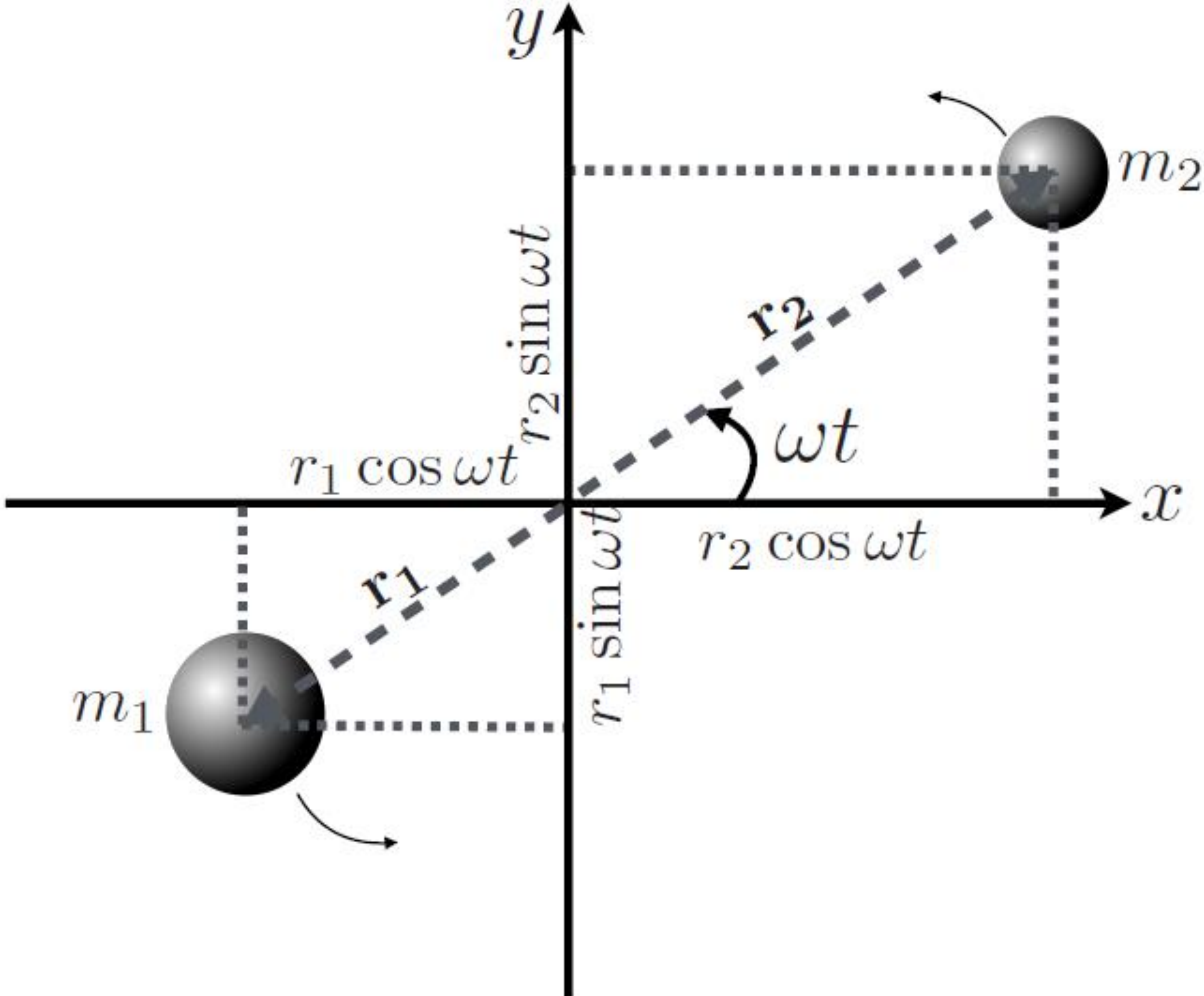


Linear fit of $f_{GW}^{-8/3}(t)$ from combined H1, L1 strain
$(\text{Frequency / Hz})^{-8/3} \cdot 10^5$
5
4
3
2
1
0
0.34
0.35
0.36
0.37
0.38
0.39
0.4
0.41
0.42
0.43
Time (s)
Frequency (Hz)
42
43
47
50
55
71
74
86
120
195


350 km
100 km
100 km
200 km
200 km



y
x
m_2
m_1
r2
r1
ωt
r_1 cos ωt
r_2 sin ωt
r_2 cos ωt
r_1 sin ωt